

①

M-III

Unit - V

Higher order partial Differential Equations

partial Differential eqn's :- A D.E which involves partial derivatives is called a partial differential eqn's

Ex:- (i) $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z$ (ii) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ are P.D.E

order of P.D.E The order of the P.D.E is the highest ordered partial derivative in the equation.

degree of P.D.E The degree of the P.D.E is the highest power of the highest ordered derivative.

Ex:- $\frac{\partial^4 u}{\partial x \partial y} = \left(\frac{\partial u}{\partial z}\right)^3$

order = 2

degree = 1

If degree of P.D.E is one then it is called Linear P.D.E.

Note If z is a function of two independent variables x and y , then we shall use the following notation for the partial derivatives of z .

$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$

In this chapter we shall consider only Linear partial differential equation of higher order with constant coefficients. It can be divided into two parts

1. Homogeneous Linear P.D.E (each term order is same, degree is one)

Ex:- $\frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + 4 \frac{\partial^2 u}{\partial y^2} = \log(x+2y)$

2. Non Homogeneous Linear P.D.E (Having different order terms, degree is one)

Ex:- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 3 \frac{\partial u}{\partial x} - 4 \frac{\partial u}{\partial y} = \sin(x+y)$

Homogeneous Linear Partial differential eqn's with constant coefficients :- An eqn's of the form

$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \dots + a_n \frac{\partial^n z}{\partial y^n} = R(x, y)$

—①

② where $a_0, a_1, a_2, \dots, a_n$ are constants, is called a homogeneous linear partial differential equation with constant coefficients of n th order.

Here all the partial derivatives are of the n th order.

Here we take $\boxed{\frac{\partial}{\partial x} = D, \quad \frac{\partial}{\partial y} = D'}$

\therefore The symbolic form of eqn (1) is

$$(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \dots + a_n D^n) z = Q(x, y)$$

$$\text{i.e., } f(D, D') z = Q(x, y)$$

Its soln consists of two parts, The Complementary soln (C.F) and the particular integral (P.I)

The Complementary function is the general soln of

$\otimes f(D, D') = 0$, it must contain n arbitrary constants where n is the order of the P.D.E.

The particular integral is a particular soln (free from arbitrary constants) of $f(D, D') = Q(x, y)$.

\therefore The complete soln of eqn (1) is $\boxed{z = C.F + P.I}$

Rules to find Complementary function.

Putting $D = m$ and $D' = 1$ in $f(D, D') = 0$, we get the auxiliary eqn (A.E) $f(m, 1) = 0$, solve it for m

(i) If the roots of A.E are m_1, m_2, \dots, m_n all are distinct (real or imaginary) then

$$C.F = f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x) + \dots + f_n(y + m_n x)$$

(ii) If two roots are equal and remaining roots are different i.e. $m_1 = m_2 = m, m_3, m_4, \dots, m_n$ then

$$C.F = f_1(y + m x) + x f_2(y + m x) + f_3(y + m_3 x) + f_4(y + m_4 x) + \dots + f_n(y + m_n x)$$

- (3) (ii) If three roots are equal and remaining roots are different i.e. $m_1 = m_2 = m_3 = m$, m_4, m_5, \dots, m_n
 Then C.F. = $f_1(y + m_1x) + x f_2(y + m_1x) + x^2 f_3(y + m_1x) + f_4(y + m_4x) + f_5(y + m_5x) + \dots + f_n(y + m_nx)$
 where $f_1, f_2, f_3, \dots, f_n$ are arbitrary functions.

Problems:

① solve $2 \frac{\partial^3 z}{\partial x^3} + 5 \frac{\partial^3 z}{\partial x \partial y} + 2 \frac{\partial^3 z}{\partial y^3} = 0$.

Sol: Since in the given P.D.E each term is same order so this is Homogeneous P.D.E.

Symbolic form of the given eqn is

$$(2 D^3 + 5 D D' + 2 D'^3) z = 0 \quad \left(\text{put } \frac{\partial}{\partial x} = D, \frac{\partial}{\partial y} = D' \right)$$

Clearly this is of the form $f(D, D') z = 0$.

where $f(D, D') = 2D^3 + 5DD' + 2D'^3$

To get the auxiliary eqn (A.E) put $D = m, D' = 1$

we have $f(m, 1) = 0$.

$$2m^3 + 5m + 2 = 0$$

$$\Rightarrow (2m+1)(m+2) = 0$$

$$\Rightarrow m_1 = -\frac{1}{2}, m_2 = -2 \text{ are different roots}$$

\therefore General soln $z = C.F$

$$= f_1(y + m_1x) + f_2(y + m_2x)$$

$$z = f_1(y - \frac{1}{2}x) + f_2(y - 2x)$$

② solve $4r + 12s + 9t = 0$

w.k.T $r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$

\therefore Given P.D.E is $4 \frac{\partial^2 z}{\partial x^2} + 12 \frac{\partial^2 z}{\partial x \partial y} + 9 \frac{\partial^2 z}{\partial y^2} = 0$

symbolic form is $(4D^2 + 12DD' + 9D'^2) z = 0$

②

Here $f(D,D') = 4D^2 + 12DD' + 9D'^2$

A.E is $f(m,1) = 0$

$\Rightarrow 4m^2 + 12m + 9 = 0$

$\Rightarrow m = \frac{-12 \pm \sqrt{12^2 - 4 \times 4 \times 9}}{2 \times 4} \quad \left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right)$

$a=4, b=12, c=9$

$\Rightarrow m = \frac{-12 \pm \sqrt{144 - 144}}{8}$

$\Rightarrow m = \frac{-3}{2}$

$\therefore m = -\frac{3}{2}, -\frac{3}{2}$ are equal roots

C.F = $f_1(y+mx) + x f_2(y+mx)$

= $f_1(y - \frac{3}{2}x) + x f_2(y - \frac{3}{2}x)$ or

C.F = $f_1(2y-3x) + x f_2(2y-3x)$

Since R.H.S of the given eqn is zero, the complete soln is $z = C.F$

$\Rightarrow z = f_1(2y-3x) + x f_2(2y-3x)$

③ solve $(D^3 - 4D^2D' + 4DD'^2)z = 0$

Here $f(D,D') = D^3 - 4D^2D' + 4DD'^2$

A.E is $f(m,1) = 0$ (ie put $D = m, D' = 1$)

$m^3 - 4m^2 + 4m = 0$

$\Rightarrow m(m^2 - 4m + 4) = 0$

$\Rightarrow m(m-2)^2 = 0$

$\therefore m = 0, 2, 2$ ie $m_1 = 0, m_2 = m_3 = 2$

Here one root is different and remaining two roots are equal roots

$\therefore C.F = f_1(y+m_1x) + f_2(y+m_2x) + x f_3(y+m_2x)$

$\Rightarrow C.F = f_1(y) + f_2(y+2x) + x f_3(y+2x)$

General soln is $z = C.F = f_1(y) + f_2(y+2x) + x f_3(y+2x)$

⑤ ④ Solve $\frac{\partial^3 z}{\partial x^3} + 2 \frac{\partial^3 z}{\partial x^2 \partial y} - \frac{\partial^3 z}{\partial x \partial y^2} - 2 \frac{\partial^3 z}{\partial y^3} = 0$.

Sol: Symbolic form of the given Homogeneous P.D.E is

$$(D^3 + 2D^2 D' - D D'^2 - 2D'^3) z = 0$$

Here $f(D, D') = D^3 + 2D^2 D' - D D'^2 - 2D'^3$

A.E is $f(m, 1) = 0$

$$\Rightarrow m^3 + 2m^2 - m - 2 = 0$$

$$m_1 = 1, m_2 = -1, m_3 = -2$$

are different roots

\therefore G.S is $z = C.F$

$$z = f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x)$$

$$\Rightarrow z = f_1(y + x) + f_2(y - x) + f_3(y - 2x)$$

$$\begin{array}{c|ccc|c} 1 & 1 & 2 & -1 & -2 \\ & 0 & 1 & 3 & 2 \\ \hline -1 & 1 & 3 & 2 & 0 \\ & 0 & -1 & -2 & -2 \\ \hline -2 & 1 & 2 & 0 & 0 \\ & 0 & -2 & 0 & 0 \\ \hline & 1 & 0 & 0 & 0 \end{array}$$

⑤ Solve $\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0$.

Sol: Symbolic form is $(D^4 - D'^4) z = 0$

Here $f(D, D') = D^4 - D'^4$

A.E is $f(m, 1) = 0$

$$\Rightarrow m^4 - 1 = 0$$

$$\Rightarrow (m^2 + 1)(m^2 - 1) = 0$$

$$\Rightarrow m^2 + 1 = 0, m^2 - 1 = 0$$

$$\Rightarrow m^2 = -1, m^2 = 1$$

$$\Rightarrow m = \pm i, m = \pm 1$$

\therefore roots are $m_1 = -i, m_2 = i, m_3 = -1, m_4 = 1$ are

General soln is $z = C.F$ different roots
 $z = f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x) + f_4(y + m_4 x)$

$$\Rightarrow z = f_1(y - ix) + f_2(y + ix) + f_3(y - x) + f_4(y + x)$$

⑥

$$\textcircled{6} \quad \frac{\partial^3 z}{\partial x^3} - 4 \frac{\partial^3 z}{\partial x^2 \partial y} + 3 \frac{\partial^3 z}{\partial x \partial y^2} = 0$$

Sol 1:- symbolic form of the given Homog P.D.E is

$$(D^3 - 4D^2D' + 3DD'^2)z = 0$$

A.E is $f(m, 1) = 0$

$$\Rightarrow m^3 - 4m^2 + 3m = 0$$

$$\Rightarrow m(m^2 - 4m + 3) = 0$$

$$\Rightarrow m(m-1)(m-3) = 0$$

$\Rightarrow m_1 = 0, m_2 = 1, m_3 = 3$ are different roots

\therefore G.S is $z = C.F = f_1(y+m_1x) + f_2(y+m_2x) + f_3(y+m_3x)$

$$\Rightarrow z = f_1(y) + f_2(y+x) + f_3(y+3x)$$

H.W

$$\textcircled{7} \quad \frac{\partial^3 z}{\partial x^3} - 7 \frac{\partial^3 z}{\partial x^2 \partial y} + 6 \frac{\partial^3 z}{\partial x \partial y^2} = 0, \text{ roots are } m = 1, 2, -3$$

$$\textcircled{2} \quad (D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0, \text{ roots are } m = 1, 2, 3$$

$$\textcircled{3} \quad (D+2D')(D-3D')^2z = 0, \text{ roots are } m = -2, 3, 3$$

$$\textcircled{7} \quad \frac{\partial^4 z}{\partial x^4} - 2 \frac{\partial^4 z}{\partial x^3 \partial y} + 2 \frac{\partial^4 z}{\partial x^2 \partial y^2} - \frac{\partial^4 z}{\partial y^4} = 0$$

Sol 1:- symbolic form of given Homog P.D.E is

$$(D^4 - 2D^3D' + 2D^2D'^2 - D'^4)z = 0 \quad (\text{put } \frac{\partial}{\partial x} = D, \frac{\partial}{\partial y} = D')$$

$$\text{Here } f(D, D') = D^4 - 2D^3D' + 2D^2D'^2 - D'^4$$

A.E is $f(m, 1) = 0$ (put $D = m, D' = 1$)

$$\Rightarrow m^4 - 2m^3 + 2m^2 - 1 = 0$$

$$\begin{array}{c|cccc|c} 1 & 1 & -2 & 2 & -1 & 1 \\ & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 0 \\ & 0 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 & 1 & 0 \\ & 0 & 1 & 1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 1 & 0 \\ & 1 & 0 & 0 & 0 & 0 \end{array}$$

$$\begin{array}{c|cccc|c} 1 & 1 & -2 & 2 & -1 & 1 \\ & 0 & 1 & -1 & -1 & 1 \\ & 0 & 1 & -1 & -1 & 1 \\ & 0 & 1 & -1 & -1 & 1 \\ & 0 & 1 & -1 & -1 & 1 \end{array}$$

G.S is $z = C.F$

$$\Rightarrow z = f_1(y+m_1x) + f_2(y+m_2x) + x f_3(y+m_2x) + x^2 f_4(y+m_2x)$$

$$\Rightarrow z = f_1(y-x) + f_2(y+x) + x f_3(y+x) + x^2 f_4(y+x)$$

roots are $m_1 = -1, m_2 = m_3 = m_4 = 1$

ie one root is different, remaining 3 roots are equal

(7) Rules for finding P.I.
particular integral of the eqn's $f(D, D') = Q(x, y)$ is

$$P.I. = \frac{1}{f(D, D')} Q(x, y)$$

where $Q(x, y)$ is a function of $ax+by$
i.e., e^{ax+by} , $\sin(ax+by)$, $\cos(ax+by)$, $\tan(ax+by)$,
 $(ax+by)^n$, $\log(ax+by)$ etc then

$$\begin{aligned} \text{Case (i)} \quad P.I. &= \frac{1}{f(D, D')} Q(ax+by) \\ &= \frac{1}{f(a, b)} \int \int \dots \int Q(u) du du \dots du \\ &\quad (n \text{ times}) \quad (n \text{ times}) \end{aligned}$$

provided $f(a, b) \neq 0$

i.e. replace D by a , D' by b where a, b are coefficients of x and y respectively in $ax+by$.

This method is applicable only when $f(a, b) \neq 0$.

If $f(a, b) = 0$, then this method fails

(2) If $f(a, b) \neq 0$, put $ax+by = u$; in $Q(ax+by)$ to get

$Q(u)$, integrate $Q(u)$ n times (re) n times

i.e. as many times as the degree of the P.D.E (Order of P.D.E)

(3) now replace u by $ax+by$ to get the required P.I.

Case (ii) Case of failure i.e. $f(a, b) = 0$

$$\text{Then } P.I. = x \cdot \frac{1}{\frac{\partial}{\partial D} [f(D, D')]} Q(ax+by)$$

i.e. multiply by x and differentiate $f(D, D')$ partially w.r. to D

If the method fails again, multiply again by x and differentiate the denominator partially w.r. to D .

⑧ Problems

① solve $\frac{\partial^3 z}{\partial x^3} - 3 \frac{\partial^3 z}{\partial x^2 \partial y} + 4 \frac{\partial^3 z}{\partial y^3} = e^{x+2y}$

clearly given P.D.E is homogeneous since each term order is same.

Symbolic form of The given P.D.E is

$$(D^3 - 3D^2D' + 4D'^3)z = e^{x+2y}$$

clearly this is of the form $f(D, D')z = Q(x, y)$

$$f(D, D') = D^3 - 3D^2D' + 4D'^3, \quad Q(x, y) = e^{x+2y}$$

A.E is $f(m, n) = 0$

$$\Rightarrow m^3 - 3m^2n + 4n^3 = 0$$

$$m_1 = -1, m_2 = m_3 = 2 = n$$

Here one root is different root and remaining two roots are equal roots

$$\begin{array}{r|rrrr|r} -1 & 1 & -3 & 0 & 4 & \\ & 0 & -1 & +4 & -4 & \\ \hline 2 & 1 & -4 & +4 & 0 & \\ & 0 & 2 & -4 & & \\ \hline 2 & 1 & & -2 & 0 & \\ & 0 & & 2 & & \\ & 1 & & 0 & & \end{array}$$

$$\therefore C.F = f_1(y+m_1x) + f_2(y+m_2x) + x f_3(y+m_3x)$$

$$\Rightarrow C.F = f_1(y-x) + f_2(y+2x) + x f_3(y+2x)$$

$$P.I = \frac{1}{f(D, D')} Q(x, y)$$

$$= \frac{1}{D^3 - 3D^2D' + 4D'^3} e^{x+2y}$$

Here $Q(x, y) = e^{x+2y}$ is of the form e^{ax+by}

so find put $D=a, D'=b$ in $f(D, D')$ we have

ie put $D=1, D'=2$ in

$$f(D, D') = D^3 - 3D^2D' + 4D'^3$$

$$\Rightarrow f(1, 2) = 1 - 3 \times 2 + 4 \times 8 = 27 \neq 0$$

$$\therefore P.I = \frac{1}{f(1, 2)} \iiint Q(u) du du du \quad \text{where } u = x+2y.$$

$$= \frac{1}{27} \iiint e^u du du du$$

order of P.D.E is 3

so, $n=3$

Integrate 3 times

(9)

$$P.I = \frac{1}{27} e^y = \frac{1}{27} e^{x+2y}$$

∴ Complete sol'n $z = C.F + P.I$

$$= z = f_1(y-x) + f_2(y+2x) + \alpha f_3(y+2x) + \frac{1}{27} e^{x+2y}$$

(2) solve $(D^3 - 4D^2D' + 4DD'^2)z = 6 \sin(3x+2y)$

Solⁿ Here $f(D, D') = D^3 - 4D^2D' + 4DD'^2$, $\phi(x, y) = 6 \sin(3x+2y)$

A.E is $f(m, n) = 0$

$$\Rightarrow m^3 - 4m^2n + 4mn^2 = 0$$

$$\Rightarrow m(m^2 - 4m + 4) = 0$$

$$\Rightarrow m(m-2)^2 = 0$$

$$\Rightarrow m = 0, 2, 2$$

i.e. $m_1 = 0, m_2 = m_3 = 2 = m$

Here one root is different root and remaining two roots are equal roots

$$C.F = f_1(y+m_1x) + f_2(y+mx) + \alpha f_3(y+mx)$$

$$C.F = f_1(y) + f_2(y+2x) + \alpha f_3(y+2x)$$

$$P.I = \frac{1}{f(D, D')} \phi(m, y)$$

$$= \frac{1}{D^3 - 4D^2D' + 4DD'^2} \cdot 6 \sin(3x+2y) \quad (\text{Compare } \sin(ax+by))$$

Here $a=3, b=2, n=\text{order}=3$
Put $D=\omega, D'=b$

$$f(D, D') = f(\omega, b) = 3^3 - 4 \times 3^2 \times 2 + 4 \times 3 \times 2^2 = 3 \neq 0$$

$$P.I = 6 \times \frac{1}{f(a, b)} \iiint \sin u \, du \, du \, du, \quad u = ax+by$$

$$= \frac{6}{3} \iiint \sin u \, du \, du \, du$$

$$= 2 \iint (-\cos u) \, du \, du$$

$$= 2 \int (-\sin u) \, du$$

$$P.I = 2 \cos u = 2 \cos(3x+2y)$$

∴ Complete sol'n $z = C.F + P.I$

$$z = f_1(y) + f_2(y+2x) + \alpha f_3(y+2x) + 2 \cos(3x+2y)$$

(10)

(3) Solve $\frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin x$

Sol:- Symbolic form of the given homo P.D.E is

$$(D^2 - 2DD' + D'^2)z = \sin(x+0y)$$

Here $f(D, D') = D^2 - 2DD' + D'^2$, $Q(x, y) = \sin(x+0y)$

A.E is $f(m, 1) = 0 \Rightarrow m^2 - 2m + 1 = 0$

$$\Rightarrow (m-1)^2 = 0$$

$\therefore m = 1, 1$ are two equal roots

$$C.F = f_1(y+mx) + x f_2(y+mx)$$

$$\therefore C.F = f_1(y+x) + x f_2(y+x)$$

$$P.I = \frac{1}{f(D, D')} Q(x, y)$$

$$= \frac{1}{D^2 - 2DD' + D'^2} \sin(x+0y) \quad \left(\begin{array}{l} \text{It is of the form} \\ \sin(ax+by) \end{array} \right)$$

put $D=a=1$, $D'=b=0$ in $f(D, D')$ we have, $n = \text{order} = 2$

$$f(D, D') = f(a, b) = f(1, 0) = 1 - 0 + 0 = 1 \neq 0$$

$$P.I = \frac{1}{f(a, b)} \iint \sin u \, du \, du, \quad u = x+0y$$

$$P.I = \frac{1}{1} \iint \sin u \, du \, du$$

$$P.I = -\sin u = -\sin x$$

$$\text{Q.S is } z = C.F + P.I = f_1(y+x) + x f_2(y+x) - \sin x$$

(4) Solve $x + y - 2z = \sqrt{2x+y}$

Sol:- Given homo P.D.E is $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 2 \frac{\partial^2 z}{\partial y^2} = \sqrt{2x+y}$

Symbolic form of given P.D.E is $(D^2 + DD' - 2D'^2)z = \sqrt{2x+y}$

Here $f(D, D') = D^2 + DD' - 2D'^2$

$$Q(x, y) = \sqrt{2x+y}$$

(11)

$$A.E \text{ is } f(m, 1) = 0$$

$$\Rightarrow m^2 + m - 2 = 0$$

$$\Rightarrow (m+2)(m-1) = 0$$

$$\Rightarrow m_1 = -2, m_2 = 1 \text{ are different roots}$$

$$C.F = f_1(y+m_1x) + f_2(y+m_2x)$$

$$\Rightarrow C.F = f_1(y-2x) + f_2(y+x)$$

$$P.I = \frac{1}{f(D, D')} Q(x, y)$$

$$= \frac{1}{D^2 + DD' - 2D^2} \sqrt{2x+y}$$

$$\text{Here } a=2, b=1, n=\frac{1}{2}, \text{ put } D=a, D'=b \text{ in } f(D, D')$$

$$\text{Now } f(D, D') = f(a, b) = f(2, 1) = 4 + 2 - 2 = 4 \neq 0$$

$$P.I = \frac{1}{f(a, b)} \iint \sqrt{u} \, du \, dv, \quad u = 2x+y$$

$$= \frac{1}{4} \iint u^{1/2} \, du \, dv$$

$$= \frac{1}{4} \int \frac{u^{3/2}}{3/2} \, dv$$

$$= \frac{1}{4} \times \frac{2}{3} \times \frac{u^{5/2}}{5/2}$$

$$P.I = \frac{1}{4} \times \frac{2}{3} \times \frac{2}{5} u^{5/2} \quad (\text{replace } u)$$

$$\Rightarrow P.I = \frac{1}{15} (2x+y)^{5/2}$$

$$\therefore \text{Complete sol'n } z = C.F + P.I$$

$$z = f_1(y-2x) + f_2(y+x) + \frac{1}{15} (2x+y)^{5/2}$$

$$(5) \text{ Solve } \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$$

Soln: Symbolic form of given Homog P.D.E is

$$(D^2 - DD') z = \sin x \cos 2y$$

$$f(D, D') = D^2 - DD' \quad Q(x, y) = \sin x \cos 2y$$

$$= \frac{1}{2} (2 \sin x \cos 2y)$$

$$= \frac{1}{2} [\sin(x+2y) + \sin(x-2y)]$$

(12)

$$A \cdot E \text{ is } -f(m, 1) = 0$$

$$\Rightarrow m^2 - m = 0$$

$$\Rightarrow m(m-1) = 0$$

$\therefore m_1 = 0, m_2 = 1$ are different roots

$$C.F = f_1(y + m_1 x) + f_2(y + m_2 x)$$

$$C.F = f_1(y) + f_2(y + x)$$

$$P.I = \frac{1}{f(D, D')} \phi(x, y)$$

$$= \frac{1}{f(D, D')} \cdot \frac{1}{2} [\sin(x+2y) + \sin(x-2y)]$$

$$= \frac{1}{2} \left[\frac{1}{f(D, D')} \sin(x+2y) + \frac{1}{f(D, D')} \sin(x-2y) \right] \quad \text{--- (1)}$$

$$\text{Now } \frac{1}{f(D, D')} \sin(x+2y) = \frac{1}{D^2 - DD'} \sin(x+2y)$$

Here $\omega = 1, b = 2, n = 2$ (order)

put $D = \omega = 1, D' = b = 2$ in $f(D, D')$ we have

$$f(D, D') = f(a, b) = f(1, 2) = 1 - 1 \times 2 = -1 \neq 0$$

$$\frac{1}{f(D, D')} \sin(x+2y) = \frac{1}{-1} \iint \sin u \, du \, dy, \quad u = x+2y$$

$$= \sin u = \sin(x+2y)$$

$$\text{By } \frac{1}{f(D, D')} \sin(x-2y) = \frac{1}{D^2 - DD'} \sin(x-2y)$$

Here $\omega = 1, b = -2, n = 2$

$$f(D, D') = f(a, b) = f(1, -2) = 1 - 1(-2) = 3 \neq 0$$

$$\frac{1}{f(D, D')} \sin(x-2y) = \frac{1}{f(a, b)} \iint \sin u \, du \, dy, \quad u = x-2y$$

$$= \frac{1}{3} \iint \sin u \, du \, dy = \frac{1}{3} \sin y$$

Sub these in (1), we have

$$= \frac{1}{3} \sin(x-2y)$$

$$P.I = \frac{1}{2} \left[\sin(x+2y) - \frac{1}{3} \sin(x-2y) \right]$$

$$\text{G.S is } \mathcal{Q} = C.F + P.I = f_1(y) + f_2(y+x) + \frac{1}{2} \sin(x+2y) - \frac{1}{6} \sin(x-2y)$$

(13)

H.W solve $(D^3 - 7DD' - 6D^3)z = \sin(x+2y) + e^{2x+y}$

$$A) z = f_1(y-x) + f_2(y+3x) + f_3(y-2x) - \frac{1}{75} \cos(x+2y) - \frac{1}{12} e^{2x+y}$$

$$(6) \text{ solve } (D^2 - 2DD' + D'^2)z = e^{x+y}$$

$$\text{Soln } f(D, D') = D^2 - 2DD' + D'^2, \quad Q(x, y) = e^{x+y}$$

$$\text{A.E is } f(m, 1) = 0$$

$$\Rightarrow m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m=1, 1 \text{ are equal roots}$$

$$\therefore \text{C.F} = f_1(y+mx) + x f_2(y+mx)$$

$$\Rightarrow \text{C.F} = f_1(y+x) + x f_2(y+x)$$

$$P.I = \frac{1}{f(D, D')} Q(x, y)$$

$$= \frac{1}{D^2 - 2DD' + D'^2} e^{x+y} \quad (\text{Compare with } e^{ax+by})$$

$$\text{Here } a=1, b=1, n=2$$

$$\text{put } D=a, D'=b \text{ in } f(D, D') \text{ we have}$$

$$f(D, D') = f(a, b) = f(1, 1) = 1^2 - 2 \times 1 \times 1 + 1^2 = 0$$

It is a case of failure,

$$\text{in this case } P.I = x \cdot \frac{1}{\frac{\partial}{\partial D} [D^2 - 2DD' + D'^2]} e^{x+y}$$

$$= x \cdot \frac{1}{2D - 2D'} e^{x+y}$$

$$\text{Here } a=1, b=1, n=1$$

$$\text{put } D=a, D'=b \text{ in } 2D - 2D' \text{ we have}$$

$$2D - 2D' = 2 \times 1 - 2 \times 1 = 0$$

again it is case of failure.

$$P.I = x^2 \cdot \frac{1}{\frac{\partial}{\partial D} (2D - 2D')} e^{x+y}$$

$$= x^2 \cdot \frac{1}{2} \cdot e^{x+y} \quad (\text{partial derivative w.r.to } D, D' \text{ is constant})$$

$$P.I = \frac{x^2}{2} e^{x+y}$$

$$\therefore \text{Ans is } z = \text{C.F} + P.I$$

(14)

⑦ solve $x + y - 6z = \cos(2x+y)$

Sol: Given $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6 \frac{\partial^2 z}{\partial y^2} = \cos(2x+y)$

Symbolic form is $(D^2 + DD' - 6D'^2)z = \cos(2x+y)$

Here $f(D, D') = D^2 + DD' - 6D'^2$, $\phi(x, y) = \cos(2x+y)$

A.E is $f(m, 1) = 0 \Rightarrow m^2 + m - 6 = 0 \Rightarrow (m+3)(m-2) = 0$

$\Rightarrow m_1 = -3, m_2 = 2$ are different roots

C.F = $f_1(y+m_1x) + f_2(y+m_2x)$

\Rightarrow C.F = $f_1(y-3x) + f_2(y+2x)$

P.I = $\frac{1}{f(D, D')} \phi(x, y)$

= $\frac{1}{D^2 + DD' - 6D'^2} \cos(2x+y)$

Here $\omega = 2, b = 1, n = 2$

put $D = \omega = 2, D' = b = 1$ in $f(D, D')$ we have

$f(D, D') = f(2, 1) = 2^2 + 2 \cdot 1 - 6 \cdot 1^2 = 0$

It is a case of failure, in this case

P.I = $x \cdot \frac{1}{\frac{\partial}{\partial D} [D^2 + DD' - 6D'^2]} \cos(2x+y)$

= $x \cdot \frac{1}{2D + D'} \cos(2x+y)$

put $D = \omega = 2, D' = b = 1, n = 1$ (order of $2D + D'$ is 1)

in the above, we have $f(2, 1) = 5 \neq 0$

P.I = $x \cdot \frac{1}{u+1} \int \cos u \, du, u = 2x+y$

= $\frac{x}{5} \sin u$

\Rightarrow P.I = $\frac{x}{5} \sin(2x+y)$

G.S is $z = \text{C.F} + \text{P.I.}$

8. Solve $(D^2 + 5DD' + 6D'^2)z = \frac{1}{y-2x}$

Sol Here $f(D, D') = D^2 + 5DD' + 6D'^2$, $Q(x, y) = \frac{1}{y-2x}$

A.E is $f(m, 1) = 0$

$\Rightarrow m^2 + 5m + 6 = 0 \Rightarrow (m+2)(m+3) = 0$

$\Rightarrow m_1 = -2, m_2 = -3$ are different roots

C.F = $f_1(y+m_1x) + f_2(y+m_2x)$

\Rightarrow C.F = $f_1(y-2x) + f_2(y-3x)$

P.I = $\frac{1}{f(D, D')} Q(x, y)$

= $\frac{1}{D^2 + 5DD' + 6D'^2} \left(\frac{1}{y-2x} \right)$ (Compare with $(ax+by)^n$)

Here $a = -2, b = 1, n = 2$

put $D = a, D' = b$ in $f(D, D')$ we have

$f(D, D') = f(a, b) = f(-2, 1) = 4 - 10 + 6 = 0$

It is a case of failure, in this case

P.I = $x \cdot \frac{1}{\frac{\partial}{\partial D} [D^2 + 5DD' + 6D'^2]} (y-2x)^{-1}$

= $x \cdot \frac{1}{2D + 5D'} (y-2x)^{-1}$

Here $a = -2, b = 1, n = 1$

put $D = a, D' = b$ in $2D + 5D'$ we have

$2D + 5D' = 2(-2) + 5(1) = 1 \neq 0$

\therefore P.I = $x \cdot \frac{1}{(1)} \int u^{-1} du, \quad u = y-2x$

= $x \int \frac{1}{u} du$

P.I = $x \log u = x \log(y-2x)$

\therefore Q.S is $z =$ C.F + P.I

$z = f_1(y-2x) + f_2(y-3x) + x \log(y-2x)$

(16)

9. Solve $4x - 4y + z = 16 \log(x+2y)$

Sol:- Given P.D.E $4 \frac{\partial^2 z}{\partial x^2} - 4 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 16 \log(x+2y)$

Symbolic form of given P.D.E is

$$(4D^2 - 4DD' + D'^2) z = 16 \log(x+2y)$$

Here $f(D, D') = 4D^2 - 4DD' + D'^2$, $Q(x, y) = 16 \log(x+2y)$

A.E is $f(m, 1) = 0 \Rightarrow 4m^2 - 4m + 1 = 0$

$\Rightarrow (2m-1)^2 = 0 \Rightarrow m = \frac{1}{2}, \frac{1}{2}$ are 2 equal roots

C.F = $f_1(y+mx) + x f_2(y+mx)$

\Rightarrow C.F = $f_1(y + \frac{1}{2}x) + x f_2(y + \frac{1}{2}x)$ (or)

C.F = $f_1(2y+x) + x f_2(2y+x)$

P.I = $\frac{1}{f(D, D')} Q(x, y)$

= $\frac{1}{4D^2 - 4DD' + D'^2} 16 \log(x+2y)$ (Compare with $\log(ax+by)$)

Here $a=1, b=2, n=2$

Sub $D=a=1, D'=b=2$ in $f(D, D')$, we have,

$f(1, 2) = 4(1)^2 - 4(1)(2) + (2)^2 = 8 - 8 = 0$

It is a case of failure, in this case

P.I = $16x \cdot \frac{1}{\frac{\partial}{\partial D} [4D^2 - 4DD' + D'^2]} \log(x+2y)$

= $16x \cdot \frac{1}{8D - 4D'} \log(x+2y)$

Put $D=a=1, D'=b=2$ in $8D - 4D'$, $n=1$

we have $8D - 4D' = 8(1) - 4(2) = 0$, Case of failure

P.I = $16x^2 \cdot \frac{1}{\frac{\partial^2}{\partial D^2} (8D - 4D')} \log(x+2y)$

= $16x^2 \cdot \frac{1}{8} \log(x+2y)$

P.I = $2x^2 \log(x+2y)$

$\therefore z = C.F + P.I$

(17) (10) Solve $(D-2D')(D-D')^3 z = e^{x+y}$

Sol:- Here $f(D,D') = (D-2D')(D-D')^3$, $Q(x,y) = e^{x+y}$

A.E is $f(m,n) = 0$

$\Rightarrow (m-2)(m-1)^3 = 0 \Rightarrow m_1 = 2, m_2 = m_3 = 1 = m$

Here one root is different root and remaining two roots are equal roots

$\therefore C.F = f_1(y+m_1x) + f_2(y+mx) + x f_3(y+mx)$

$\Rightarrow C.F = f_1(y+2x) + f_2(y+x) + x f_3(y+x)$

P.F = $\frac{1}{f(D,D')} Q(x,y)$

$= \frac{1}{(D-D')^3(D-2D')} e^{x+y}$

$= \frac{1}{(D-D')^3} \left\{ \frac{1}{D-2D'} e^{x+y} \right\}$

$= \frac{1}{(D-D')^3} \left\{ \frac{1}{1-2} \int e^u du \right\}$ (anant by $\because a=1, b=1, n=1$)

$= \frac{1}{(D-D')^3} \left\{ -e^u \right\}$

$= \frac{-1}{(D-D')^3} e^{x+y}$

$= \frac{-1}{0} e^{x+y}$, Case of failure put $D=1, D'=1$

$= -x \cdot \frac{1}{\frac{\partial}{\partial D} [(D-D')^3]} e^{x+y}$

$= -x \cdot \frac{1}{2(D-D')} e^{x+y}$

$= -\frac{x}{2} \cdot \frac{1}{(1-1)} e^{x+y}$ again case of failure

$P.F = -\frac{x}{2} \left\{ x \cdot \frac{1}{\frac{\partial}{\partial D} (D-D')} e^{x+y} \right\} = -\frac{x^2}{2} e^{x+y}$

$\therefore Q2 C.F + P.F = f_1(y+2x) + f_2(y+x) + x f_3(y+x) - \frac{x^2}{2} e^{x+y}$

P.E of $f(D, D') z = Q(x, y)$ when $Q(x, y) = x^m y^n$

when $Q(x, y) = x^m y^n$ then

$$P.E = \frac{1}{f(D, D')} Q(x, y)$$

$$= \frac{1}{f(D, D')} x^m y^n$$

$$P.E = [f(D, D')]^{-1} x^m y^n$$

If $m < n$, expand $[f(D, D')]^{-1}$ in powers of $\frac{D}{D'}$

i.e. take common D' from $f(D, D')$

If $n < m$, expand $[f(D, D')]^{-1}$ in powers of $\frac{D'}{D}$

i.e. take common D from $f(D, D')$

Also we have $\frac{1}{D} Q(x, y) = \int Q(x, y) dx$
 $y \text{ constant}$

$$\frac{1}{D'} Q(x, y) = \int Q(x, y) dy$$

$x \text{ constant}$

Note (i) $Q(x, y) = x^m y^n$, if x power is highest then take common D from $f(D, D')$

(ii) If y power is highest then take common D' from $f(D, D')$.

Problems Q solve $\frac{\partial^3 z}{\partial x^3} - 2 \frac{\partial^3 z}{\partial x \partial y^2} = 2e^{2x} + 3x^2 y$

Sol 1 symbolic form of the given eqn is put $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$

$$(D^3 - 2D D'^2) z = 2e^{2x} + 3x^2 y$$

Here $f(D, D') = D^3 - 2D D'^2$, $Q(x, y) = 2e^{2x} + 3x^2 y$

A.E is $f(m, n) = 0$

$$\Rightarrow m^3 - 2m n^2 = 0$$

$$\Rightarrow m^2 (m - 2n^2) = 0$$

$$\Rightarrow m_1 = 2, m_2 = m_3 = 0$$

Here one root is different and remaining two roots are equal.

$$C.F = f_1(y + m_1 x) + f_2(y + m_2 x) + x f_3(y + m_3 x)$$

19

$$C.F = f_1(y+2x) + f_2(y) + x f_3(y).$$

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{f(D,D')} (2e^{2x} + 3x^2 y)$$

$$= \left[2 \cdot \frac{1}{f(D,D')} e^{2x} + 3 \frac{1}{f(D,D')} x^2 y \right]$$

$$P.I = 2P_1 + 3P_2 \quad \text{--- (1)}$$

$$\text{Now } 2P_1 = 2 \cdot \frac{1}{f(D,D')} e^{2x}$$

$$= 2 \cdot \frac{1}{D^3 - 2DD'} e^{2x+0y} \quad (\text{Compare with } e^{ax+by})$$

Here $a=2, b=0, m=3$ (order)

put $D=a=2, D'=b=0$ in $f(D,D')$, we have

$$f(D,D') = f(2,0) = 2^3 - 2 \times 2^2 \times 0 = 8 \neq 0$$

$$\therefore 2P_1 = 2 \cdot \frac{1}{8} \iint e^u du dv du, \quad u=2x+0y$$

$$= \frac{1}{4} e^u \Rightarrow 2P_1 = \frac{1}{4} e^{2x}$$

$$3P_2 = 3 \frac{1}{f(D,D')} x^2 y$$

$$= 3 \cdot \frac{1}{D^3 - 2DD'} x^2 y \quad (\text{Compare with } x^m y^n)$$

$m=2, n=1, m>n$ (ie x power is high).

So take common $x^{\text{highest power of } D}$ from $f(D,D')$, we have

$$= 3 \cdot \frac{1}{D^3 \left[1 - \frac{2D'}{D} \right]} x^2 y$$

$$= 3 \cdot \frac{1}{D^3} \left[1 - \frac{2D'}{D} \right]^{-1} x^2 y$$

Q20

$$2P_2 = \frac{3}{D^3} \left[1 + \frac{2D}{D} + \left(\frac{2D}{D}\right)^2 + \dots \right] x^2 y$$

$$\because (1-x)^{-1} = 1 + x + x^2 + \dots$$

$$= \frac{3}{D^3} \left[1 + \frac{2D}{D} \right] x^2 y \quad (\text{neglect higher power from } D^2 \text{ and higher power})$$

$$= \frac{3}{D^3} \left[x^2 y + \frac{2D}{D} (x^2 y) \right]$$

Since y power is one so consider upto D power is one

$$= \frac{3}{D^3} \left[x^2 y + \frac{2}{D} x^2 D(y) \right]$$

$Dy = 1$, $D^2 y = D^3 y = \dots = 0$
 \rightarrow w.r.to D, D is constant

$$= \frac{3}{D^3} \left[x^2 y + \frac{2}{D} x^2 \right]$$

$$= \frac{3}{D^3} \left[x^2 y + 2 \cdot \frac{x^3}{3} \right] \quad \left(\frac{1}{D} \rightarrow \int dx \right)$$

$$= 3y \iiint x^2 dx dx dx + 2 \iiint x^3 dx dx dx$$

$$= 3y \cdot \left(\frac{x^5}{60} \right) + 2 \cdot \left(\frac{x^6}{120} \right)$$

$$2P_2 = y \frac{x^5}{20} + \frac{x^6}{60}$$

Sub $2P_1$, $2P_2$ in (1), we have

$$P.D = \frac{1}{u} e^{2x} + y \frac{x^5}{20} + \frac{x^6}{60}$$

\therefore Q.S is $Q = C.F + P.D$

$$Q = f_1(y+2x) + f_2(y) + x f_3(y) + \frac{1}{u} e^{2x} + y \frac{x^5}{20} + \frac{x^6}{60}$$

Note

$$\frac{1}{D^3} \rightarrow \iiint dx dx dx$$

$$\text{likewise } \frac{1}{D^3} \rightarrow \iiint dy dy dy$$

$$\frac{1}{D^2} \rightarrow \iint dx dx$$

$$\frac{1}{D^2} \rightarrow \iint dy dy$$

(21)

② solve $(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy$.

Sol:- Here $f(D,D') = D^2 - 6DD' + 9D'^2$, $Q(x,y) = 12x^2 + 36xy$

A.E is $f(m,1) = 0$ (put $D=m, D'=1$)

$\Rightarrow m^2 - 6m + 9 = 0$

$\Rightarrow (m-3)^2 = 0$

$\Rightarrow m=3, 3$ are two equal roots

$\therefore C.F = f_1(y+mx) + x f_2(y+mx)$

$\Rightarrow C.F = f_1(y+3x) + x f_2(y+3x)$

P.I = $\frac{1}{f(D,D')} Q(x,y)$

$= \frac{1}{f(D,D')} [12x^2 + 36xy] = 12 \cdot \frac{1}{f(D,D')} x^2 + 36 \cdot \frac{1}{f(D,D')} xy$

$2P_1 = 12 \cdot \frac{1}{f(D,D')} x^2 \quad \quad \quad = 2P_1 + 2P_2 \quad \text{--- (1)}$

$= 12 \cdot \frac{1}{D^2 - 6DD' + 9D'^2} (x+0y)^2 \quad (\text{compare with } (ax+by)^2)$

$a=1, b=0, \quad n = \text{order of D.E} = 2$

put $D=a=1, D'=b=0$ in $f(D,D')$, we have

$f(D,D') = f(1,0) = 1 \neq 0$

$2P_1 = 12 \cdot \frac{1}{1} \int \int u^2 du dv \quad (u=x)$

$= 12 \cdot \frac{u^4}{12} \Rightarrow 2P_1 = (x^4) = x^4$

$2P_2 = 36 \cdot \frac{1}{f(D,D')} xy$

$= 36 \cdot \frac{1}{D^2 - 6DD' + 9D'^2} (xy) \quad \text{Compare } (x^m y^n)$

$= 36 \cdot \frac{1}{(D-3D')^2} (xy)$
Here $m=1, n=1$ so $m=n$.

Hence take higher power of D or higher power of D' common from $f(D,D')$

Here I take highest power of D common from $f(D,D')$

(22)

$$\partial P_2 = 36 \cdot \frac{1}{D^{\sqrt{}} \left[1 - \frac{3D}{D} \right]^{\sqrt{}}} xy$$

$$= \frac{36}{D^{\sqrt{}}} \left[1 - \frac{3D}{D} \right]^{\sqrt{}} (xy)$$

$$= \frac{36}{D^{\sqrt{}}} \left[1 + 2 \left(\frac{3D}{D} \right) \right] (xy) \quad ((1-x)^{-2} = 1 + 2x + 3x^2 + \dots)$$

y power is one so
consider upto D power is one

$$= \frac{36}{D^{\sqrt{}}} \left[xy + \frac{6}{D} D^1(xy) \right]$$

$$= \frac{36}{D^{\sqrt{}}} \left[xy + \frac{6}{D} x \right]$$

$$= 36y \frac{1}{D^{\sqrt{}}}(x) + 36 \times 6 \frac{1}{D^3}(x)$$

$$= 36y \int x dx dx + 36 \times 6 \int \int x dx dx dx$$

$$= 36y \left(\frac{x^3}{6} \right) + 36 \times 6 \left(\frac{x^4}{4} \right)$$

$$\partial P_2 = 6x^3y + 9x^4$$

Sub $\partial P_1, \partial P_2$ in (1), we have

$$P.I = x^4 + 6x^3y + 9x^4 = 6x^3y + 10x^4$$

$$\therefore \text{G.S is } Q = C.F + P.I$$

$$x Q = f_1(y+3x) + x f_2(y+3x) + 6x^3y + 10x^4$$

(23)

$$\frac{\partial^2 Q}{\partial x^{\sqrt{}}} - \frac{\partial^2 Q}{\partial y^{\sqrt{}}} = x^{\sqrt{}} + y^{\sqrt{}}$$

sol.

Symbolic form of the given D.E is $(D^{\sqrt{}} - D^{\sqrt{}})Q = x^{\sqrt{}} + y^{\sqrt{}}$

Here $f(D, D^1) = D^{\sqrt{}} - D^{\sqrt{}}$, $Q(x, y) = x^{\sqrt{}} + y^{\sqrt{}}$

$$\text{A.E is } f(m, D) = 0 \Rightarrow m^{\sqrt{}} - 1 = 0$$

$$\Rightarrow m = \pm 1 \text{ i.e. } m_1 = -1, m_2 = 1$$

are different roots

(23)

$$C \cdot F = f_1(y+m_1x) + f_2(y+m_2x)$$

$$\Rightarrow C \cdot F = f_1(y-x) + f_2(y+x)$$

$$P \cdot I = \frac{1}{f(D,D)} \phi(x,y)$$

$$= \frac{1}{D^2 - D_1^2} (x^2 + y^2)$$

$$= \frac{1}{D^2 - D_1^2} (x^2) + \frac{1}{D^2 - D_1^2} (y^2) = \cancel{\frac{1}{2}} \partial P_1 + \cancel{\frac{1}{2}} \partial P_2 \quad \text{--- (1)}$$

$$\partial P_1 = \frac{1}{D^2 - D_1^2} (x^2) = \frac{1}{D^2 - D_1^2} (x+0y)^2 \quad \text{(Compare } (ax+by)^2)$$

$$a=1, b=0, n=2$$

$$\text{put } D=a=1, D_1=b=0 \text{ in } f(D,D)$$

$$f(D,D) = f(1,0) = 1 \neq 0$$

$$\partial P_1 = \frac{1}{1} \iint u^2 du dv \quad \text{where } u=x+0y$$

$$= \frac{u^4}{4} = \frac{x^4}{4}$$

$$\partial P_2 = \frac{1}{D^2 - D_1^2} (y^2) = \frac{1}{D^2 - D_1^2} (0x+y)^2$$

$$a=0, b=1, n=2 \text{ (order of } D, D_1)$$

$$f(D,D) = f(0,1) = 0^2 - 1^2 = -1 \neq 0$$

$$\partial P_2 = \frac{1}{-1} \iint u^2 du dv, \text{ where } u=0x+y$$

$$= -\frac{u^4}{4} = -\frac{y^4}{4}$$

$$\iint u^2 du dv = \int \left[\frac{u^3}{3} \right] dv$$

$$= \int \left(\frac{y^3}{3} \right) dy$$

$$= \frac{y^4}{12}$$

sub $\partial P_1, \partial P_2$ in (1), we have

$$P \cdot I = \frac{x^4}{4} - \frac{y^4}{4}$$

$$\therefore G.S \text{ is } \partial = C \cdot F + P \cdot I$$

$$\therefore \partial = f_1(y-x) + f_2(y+x) + \frac{1}{4} (x^4 - y^4)$$

$$(4) \quad \frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2$$

Sol: Symbolic form of the given D.E is

$$(D^2 + 2DD' + D'^2)z = x^2 + xy + y^2$$

Here $f(D, D') = D^2 + 2DD' + D'^2$, $Q(x, y) = x^2 + xy + y^2$

A.E is $f(m, 1) = 0 \Rightarrow m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0$

$\Rightarrow m = -1, -1$ are two equal roots

C.F = $f_1(y+mx) + x f_2(y+mx)$

\therefore C.F = $f_1(y-x) + x f_2(y-x)$

P.I = $\frac{1}{f(D, D')} Q(x, y)$

$= \frac{1}{f(D, D')} (x^2 + xy + y^2)$

$= \frac{1}{f(D, D')} x^2 + \frac{1}{f(D, D')} (xy) + \frac{1}{f(D, D')} (y^2)$

$= \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3$ (A)

$\mathcal{P}_1 = \frac{1}{f(D, D')} x^2 = \frac{1}{(D+D')^2} x^2$

$= \frac{1}{(D+D')^2} (x+0y)^2$ (Compare with $ax+by$)

$a=1, b=0, n=2$

$f(D, D') = f(a, b) = f(1, 0) = (1+0)^2 = 1 \neq 0$

$\mathcal{P}_1 = \frac{1}{1} \iint u^2 du dv$, where $u = x+0y$

$= \frac{u^4}{12} = \frac{x^4}{12}$

$\mathcal{P}_3 = \frac{1}{f(D, D')} y^2 = \frac{1}{(D+D')^2} (0x+y)^2$

$a=0, b=1, n=2$

$f(D, D') = f(a, b) = (0+1)^2 = 1 \neq 0$

$\mathcal{P}_3 = \frac{1}{1} \iint u^2 du dv$, $u = 0x+y$

$= \frac{u^4}{12} = \frac{y^4}{12}$

(25)

$$\partial P_2 = \frac{1}{f(D,D')} (xy)$$

$$= \frac{1}{(D+D')^2} (xy) \quad (\text{Compare with } x^m y^n, \text{ here } m=n=1$$

so take highest power of D

or take highest power of D'
Common from $f(D,D')$)

$$= \frac{1}{D^2 \left[1 + \frac{D'}{D}\right]^2} (xy)$$

$$= \frac{1}{D^2} \left[1 + \frac{D'}{D}\right]^{-2} (xy)$$

$$= \frac{1}{D^2} \left[1 - 2\frac{D'}{D}\right] (xy)$$

$$= \frac{1}{D^2} \left[xy - 2\frac{D'}{D}(xy)\right]$$

$$= \frac{1}{D^2} \left[xy - \frac{2}{D}(x)D(y)\right]$$

$$= y \frac{1}{D^2}(x) - \frac{2}{D^3}(x)$$

$$= y \iint x dx dx - 2 \iiint x dx dx dx$$

$$\partial P_2 = y \frac{x^3}{6} - 2 \left(\frac{x^4}{24}\right)$$

$$\partial P_2 = \frac{yx^3}{6} - \frac{x^4}{12}$$

Sub $\partial P_1, \partial P_2, \partial P_3$ value in (1), we have

$$P \cdot D = \frac{x^4}{12} + \frac{yx^3}{6} - \frac{x^4}{12} + \frac{y^4}{12}$$

$$P \cdot D = \frac{yx^3}{6} + \frac{y^4}{12}$$

$$\text{G.S is } Q = C \cdot P + P \cdot D$$

$$= f_1(4-x) + x f_2(4-x) + \frac{yx^3}{6} + \frac{y^4}{12}$$

Ans (1) $\frac{\partial^2 Q}{\partial x^2} + 3\frac{\partial^2 Q}{\partial x \partial y} + 2\frac{\partial^2 Q}{\partial y^2} = 12xy$ A) $f_1(4-x) + f_2(4-2x) + 2x^3y - \frac{3}{2}x^4$

(2) $(D^2 - 2DD')Q = e^{2x} + x^3y$ A) $f_1(y) + f_2(y+2x) + \frac{1}{4}e^{2x} + \frac{x^3y}{20} + \frac{y^6}{60}$

(26)

(5)

$$\frac{\partial^3 q}{\partial x^3} + \frac{\partial^3 q}{\partial y^3} = x^3 y \checkmark$$

Solve

Here symbolic soln of given D.E is $(D^3 - D^3) q = x^3 y \checkmark$

$$\text{Here } f(D, D) = D^3 - D^3, \quad Q(x, y) = x^3 y \checkmark$$

$$\text{A.E is } f(m, i) = 0$$

$$\Rightarrow m^3 - 1 = 0$$

$$\Rightarrow (m-1)(m^2+m+1) = 0 \Rightarrow (m-1) = 0, m^2+m+1 = 0$$

$$\Rightarrow m_1 = 1, \quad m_2 = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times 1}}{2}$$

$$a=1, b=1, c=1$$

$$m_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-1 \pm \sqrt{-3}}{2}$$

$$\Rightarrow m_2 = \frac{-1 \pm i\sqrt{3}}{2}$$

$$\therefore m_1 = 1, \quad m_2 = \frac{-1+i\sqrt{3}}{2}, \quad m_3 = \frac{-1-i\sqrt{3}}{2} \text{ are different roots}$$

$$\text{Let us take } m_1 = 1, \quad m_2 = \frac{-1+i\sqrt{3}}{2} = \omega, \quad m_3 = \frac{-1-i\sqrt{3}}{2} = \omega^2$$

Since we know the given eqn is $m^3 - 1 = 0 \Rightarrow m^3 = 1$ means cube roots of unity which are $1, \omega, \omega^2$

$$\therefore \text{C.F} = f_1(y+m_1x) + f_2(y+m_2x) + f_3(y+m_3x)$$

$$\Rightarrow \text{C.F} = f_1(y+x) + f_2(y+\omega x) + f_3(y+\omega^2 x)$$

$$P.I = \frac{1}{f(D, D)} Q(x, y)$$

$$= \frac{1}{f(D^3 - D^3)} x^3 y \quad (x^m y^n, \quad m \geq 3, n \geq 2)$$

$$= \frac{1}{f(D^3 - D^3)} \quad m > n \text{ so take common highest power of } D \text{ from } f(D, D) \text{ we have}$$

$$= \frac{1}{D^3 \left[1 - \frac{D^3}{D^3} \right]} x^3 y$$

$$= \frac{1}{D^3} \left[1 - \frac{D^3}{D^3} \right]^{-1} x^3 y$$

$$= \frac{1}{D^3} \left[1 + \frac{D^3}{D^3} + \left(\frac{D^3}{D^3} \right)^2 + \dots \right] x^3 y$$

$$(27) \quad P.I = \frac{1}{D^3} (x^3 y^3) = y^3 \iiint x^3 dx dx dx.$$

$$P.I = \frac{y^3 \cdot x^6}{120} \quad \int \frac{x^3}{4} \quad \frac{x^5}{20} \quad \frac{x^6}{120}.$$

$$\therefore \text{G.S is } y = C.F + P.I$$

$$x \quad z = f_1(y+1) + f_2(y+\omega y) + f_3(y+\omega^2 y) + \frac{y^3 x^6}{120}.$$

General method for finding P.I.

If $Q(x, y)$ is of a form different from the forms already discussed, Then use.

$$\frac{1}{D-mD'} Q(x, y) = \int Q(x, C-mx) dx$$

ie replace y by $C-mx$, and after integration replace C by $y+mx$.

procedure factorise $f(D, D')$ into linear factors, then

$$P.I = \frac{1}{f(D, D')} Q(x, y)$$

$$= \frac{1}{(D-m_1 D')(D-m_2 D') \dots} Q(x, y)$$

$$= \frac{1}{D-m_1 D'} \cdot \frac{1}{D-m_2 D'} \dots Q(x, y)$$

Repeated application of the above rule gives the P.I

Problem (1) solve $(D^2 - DD' - 2D^2)z = (y-1)e^x$.

Sol $f(D, D') = D^2 - DD' - 2D^2$, $Q(x, y) = (y-1)e^x$

$$\text{A.E is } f(m, 1) = 0 \Rightarrow m^2 - m - 2 = 0$$

$$\Rightarrow (m-2)(m+1) = 0$$

$\Rightarrow m_1 = 2, m_2 = -1$ are different roots

$$C.F = f_1(y+m_1 x) + f_2(y+m_2 x)$$

$$= f_1(y+2x) + f_2(y-x)$$

$$P-I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{D^2 - D - 2D'} (y-1)e^x$$

$$= \frac{1}{(D-2D')(D+D')} (y-1)e^x$$

$$= \frac{1}{D-2D'} \cdot \frac{1}{D+D'} (y-1)e^x$$

(Comparing $D+D'$ with

$$D-mD', m=1$$

replace $y = c - mx$

$$= \frac{1}{D-2D'} \int \frac{e^x}{v} (c+x-1) dx$$

$$\begin{aligned} y &= c+x \\ \Rightarrow c &= y-x \end{aligned}$$

$$= \frac{1}{D-2D'} \left\{ u \int v dx - \left(\frac{du}{dx} \int v dx \right) dx \right\}$$

$$= \frac{1}{D-2D'} \left\{ (c+x-1) \int e^x dx - \left[\frac{d}{dx} (c+x-1) \int e^x dx \right] dx \right\}$$

$$= \frac{1}{D-2D'} \left\{ (c+x-1) e^x - \int 1 \times e^x dx \right\}$$

$$= \frac{1}{D-2D'} \left\{ (c+x-1) e^x - e^x \right\}$$

replace $c = y-x$, we have

$$= \frac{1}{D-2D'} \left\{ (y-x+x-1) e^x - e^x \right\}$$

$$= \frac{1}{D-2D'} \left[(y-1) e^x - e^x \right]$$

$$= \frac{1}{D-2D'} \left[(y-2) e^x \right]$$

Compare $D-2D'$ with $D-mD'$, here $m=2$

replace y by $c - mx = c - 2x$.

after integration replace $c = y + 2x$.

(29)

$$P.I = \int \underbrace{(C-2x-2)}_u \underbrace{e^x}_v dx$$

$$= u \int v dx - \int \left(\frac{du}{dx} \int v dx \right) dx$$

$$= (C-2x-2) \int e^x dx - \int \left[\frac{d}{dx} (C-2x-2) \int e^x dx \right] dx$$

$$= (C-2x-2) e^x - \int (-2) e^x dx$$

$$= (C-2x-2) e^x + 2e^x$$

$$P.I = (C-2x-2+2) e^x$$

$$= (C-2x) e^x$$

$$P.I = (y+2x-2x) e^x \quad (\because C = y+2x)$$

$$\Rightarrow P.I = y e^x$$

$$\therefore G.S \& = C.F + P.I$$

$$\underline{\underline{z = f_1(y+2x) + f_2(y-x) + y e^x}}$$

(2) solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6 \frac{\partial^2 z}{\partial y^2} = y \cos x$.

Sol: Symbolic form of the given eqn is

$$(D^2 + DD' - 6D'^2) z = y \cos x$$

Here $f(D, D') = D^2 + DD' - 6D'^2$, $Q(x) = y \cos x$.

A.E is $f(m, 1) = 0 \Rightarrow m^2 + m - 6 = 0$

$$\Rightarrow (m+3)(m-2) = 0$$

$\therefore m_1 = -3, m_2 = 2$ are diff roots

$$\therefore C.F = f_1(y+m_1x) + f_2(y+m_2x)$$

$$\Rightarrow C.F = f_1(y-3x) + f_2(y+2x)$$

$$P.I = \frac{1}{f(D, D')} Q(x, y)$$

$$= \frac{1}{D^2 + DD' - 6D'^2} (y \cos x)$$

30

$$P.D = \frac{1}{(D+3D')(D-2D')} (y \cos x)$$

$$= \frac{1}{D+3D'} \left\{ \frac{1}{D-2D'} (y \cos x) \right\}$$

Compare $D-2D'$ with $D-mD'$,
Here $m=2$
Put $y = C - mx$
 $\Rightarrow y = C - 2x$
replace after integration
 $\boxed{C = y + 2x}$

$$= \frac{1}{D+3D'} \int \underbrace{(C-2x)}_u \underbrace{\cos x}_{v} dx$$

$$= \frac{1}{D+3D'} \left\{ u \int v dx - \int \frac{du}{dx} \int v dx \right\}$$

$$= \frac{1}{D+3D'} \left\{ (C-2x) \int \cos x dx - \int \frac{d}{dx} (C-2x) \int \cos x dx \right\}$$

$$= \frac{1}{D+3D'} \left\{ (C-2x) \sin x - \int (-2) \sin x dx \right\}$$

$$= \frac{1}{D+3D'} \left\{ (C-2x) \sin x - 2 \cos x \right\}$$

$$= \frac{1}{D+3D'} \left\{ (y+2x-2x) \sin x - 2 \cos x \right\}$$

$$= \frac{1}{D-(-3D')} (y \sin x - 2 \cos x)$$

Here $m=-3$,
Put $y = C - mx$
 $\Rightarrow y = C + 3x$
 $\Rightarrow \boxed{C = y - 3x}$

$$= \int [(C+3x) \sin x - 2 \cos x] dx$$

$$= \int \underbrace{(C+3x)}_u \underbrace{\sin x}_{v} dx - 2 \int \cos x dx$$

$$= u \int v dx - \int \frac{du}{dx} \int v dx - 2 \sin x$$

$$= (C+3x) \int \sin x dx - \int \frac{d}{dx} (C+3x) \int \sin x dx - 2 \sin x$$

$$= -(C+3x) \cos x - \int 3 (-\cos x) dx - 2 \sin x$$

$$= -(C+3x) \cos x + 3 \sin x - 2 \sin x$$

$$= -[y-3x+3x] \cos x + \sin x$$

P.D = $-y \cos x + \sin x$

$\Delta = C.F + P.D$

(31) (3) solve $(D^2 + DD' - 6D'^2)z = x^y \sin(x+y)$

Sol. Here $f(D, D') = D^2 + DD' - 6D'^2$, $\omega(x, y) = x^y \sin(x+y)$

A.E. if $f(m, n) = 0 \Rightarrow m^2 + m - 6 = 0$
 $\Rightarrow (m+3)(m-2) = 0$

$\Rightarrow m_1 = -3, m_2 = 2$ are diff roots

C.F. = $f_1(y + m_1 x) + f_2(y + m_2 x)$

\therefore C.F. = $f_1(y - 3x) + f_2(y + 2x)$

P.I. = $\frac{1}{f(D, D')} \omega(x, y)$

= $\frac{1}{(D+3D')(D-2D')} x^y \sin(x+y)$

= $\frac{1}{D+3D'} \left\{ \frac{1}{D-2D'} x^y \sin(x+y) \right\}$

Here $m = 2$,

Put $y = C - mx$

$\therefore y = C - 2x$

$\therefore \boxed{C = y + 2x}$

= $\frac{1}{D+3D'} \int x^y \sin(x + C - 2x) dx$

= $\frac{1}{D+3D'} \int \underbrace{x^y}_u \sin \underbrace{(C-x)}_v dx$

= $\frac{1}{D+3D'} [u v_1 - u' v_2 + u'' v_3]$ (Bernoulli's rule for $\int u v dx$)

$u = x^y$

$v = \sin(C-x)$

$u' = 2x$

$u'' = 2$

$v_1 = \int v dx = -\frac{\cos(C-x)}{-1} = \cos(C-x)$

$v_2 = \int v_1 dx = \frac{\sin(C-x)}{-1} = -\sin(C-x)$

$v_3 = \int v_2 dx = -\left[\frac{-\cos(C-x)}{-1} \right] = -\cos(C-x)$

P.I. = $\frac{1}{D+3D'} \left\{ x^y \cos(C-x) - 2x [-\sin(C-x)] + 2 [-\cos(C-x)] \right\}$

= $\frac{1}{D+3D'} \left\{ x^y \cos(C-x) + 2x \sin(C-x) - 2 \cos(C-x) \right\}$

= $\frac{1}{D+3D'} \left\{ x^y \cos(y+2x-x) + 2x \sin(y+2x-x) - 2 \cos(y+2x-x) \right\}$

($\because C = y + 2x$)

(32)

$$P.I = \frac{1}{D+3D} \{ x^3 \cos(y+x) + 2x \sin(y+x) - 2 \cos(y+x) \}$$

Compare $D+3D$ Here $m=-3$, Put $y = C - mx$
with $D-mD$ $\Rightarrow y = C + 3x$

$$\Rightarrow \boxed{C = y - 3x}$$

$$P.I = \int x^3 \cos(y-3x+x) + 2x \sin(y-3x+x)$$

$$P.I = \int [x^3 \cos(C+3x+x) + 2x \sin(C+3x+x) - 2 \cos(C+3x+x)] dx$$

$$P.I = \int x^3 \cos(C+4x) dx + 2 \int x \sin(C+4x) dx - 2 \int \cos(C+4x) dx \quad \text{--- (1)}$$

$$\int \frac{x^3 \cos(4x+C)}{1} dx = u v_1 - u' v_2 + u'' v_3$$

$$u = x^3 \quad v = \cos(4x+C)$$

$$u' = 3x^2 \quad v_1 = \int v dx = \frac{\sin(4x+C)}{4}$$

$$u'' = 6x \quad v_2 = \int v_1 dx = -\frac{\cos(4x+C)}{4^2} = -\frac{\cos(4x+C)}{16}$$

$$v_3 = \int v_2 dx = -\frac{\sin(4x+C)}{4^3} = -\frac{\sin(4x+C)}{64}$$

$$\begin{aligned} \int x^3 \cos(4x+C) dx &= x^3 \frac{\sin(4x+C)}{4} - 3x^2 \left[-\frac{\cos(4x+C)}{16} \right] + 6x \left[-\frac{\sin(4x+C)}{64} \right] \\ &= \frac{x^3}{4} \sin(4x+y-3x) + \frac{3x^2}{8} \cos(4x+y-3x) - \frac{3x}{32} \sin(4x+y-3x) \end{aligned}$$

$$\int x^3 \cos(4x+C) dx = \frac{x^3}{4} \sin(x+y) + \frac{3x^2}{8} \cos(x+y) - \frac{3x}{32} \sin(x+y)$$

$$\int \cos(C+4x) dx = \frac{\sin(4x+C)}{4} = \frac{1}{4} \sin(4x+y-3x) = \frac{1}{4} \sin(x+y)$$

$$\int x \sin(4x+C) dx = u v_1 - u' v_2 \quad \begin{aligned} u &= x & v &= \sin(4x+C) \\ u' &= 1 & v_1 &= \int v dx = -\frac{\cos(4x+C)}{4} \\ & & v_2 &= \int v_1 dx = -\frac{\sin(4x+C)}{16} \end{aligned}$$

(33)

$$\int x \sin(4x+c) dx = x \left[\frac{-\cos(4x+c)}{4} \right] - \left[\frac{-\sin(4x+c)}{16} \right]$$

$$= -\frac{x}{4} \cos(4x+c-3\pi) + \frac{1}{16} \sin(4x+c-3\pi)$$

$$\int x \sin(4x+c) dx = \frac{-x}{4} \cos(x+y) + \frac{1}{16} \sin(x+y)$$

sub all these values in (1), we have

$$\begin{aligned} P.D. &= \frac{x^2}{4} \sin(x+y) + \frac{x}{8} \cos(x+y) - \frac{1}{32} \sin(x+y) + \\ &2 \left[-\frac{x}{4} \cos(x+y) + \frac{1}{16} \sin(x+y) \right] - 2 \left[\frac{1}{4} \sin(x+y) \right] \\ &= \frac{x^2}{4} \sin(x+y) + \frac{x}{8} \cos(x+y) - \frac{1}{32} \sin(x+y) - \frac{x}{2} \cos(x+y) \\ &\quad + \frac{1}{8} \sin(x+y) - \frac{1}{2} \sin(x+y) \\ &= \sin(x+y) \left[\frac{x^2}{4} - \frac{1}{32} + \frac{1}{8} - \frac{1}{2} \right] + \cos(x+y) \left[\frac{x}{8} - \frac{x}{2} \right] \\ &= \sin(x+y) \left[\frac{x^2}{4} + \frac{-1+4-16}{32} \right] + x \cos(x+y) \left[\frac{1-4}{8} \right] \end{aligned}$$

$$P.D. = \left(\frac{x^2}{4} - \frac{13}{32} \right) \sin(x+y) - \frac{3x}{8} \cos(x+y)$$

\therefore G.S is $Q = C.F + P.D$

$$= f_1(y-3x) + f_2(y+2x) + \left(\frac{x^2}{4} - \frac{13}{32} \right) \sin(x+y) - \frac{3x}{8} \cos(x+y).$$

(4) - solve r-s at H.W problem

(1) $(D^2 + DD' - 6D'^2)z = y \sin x$ A) $f_1(y+2x) + f_2(y-3x) - y \sin x - \cos x$

(2) $(D-D')(D+2D')z = (y+1)e^x$ A) $f_1(y+x) + f_2(y-2x) + ye^x$

(3) $(D^2 + 2DD' + D'^2)z = 2 \sin y - x \cos y$

(34)

Non-homogeneous Linear partial differential eqn's

In the eqn $f(D, D')z = Q(x, y)$, if the polynomial $f(D, D')$ in D, D' is not homogeneous (ie having different powers/orders) Then it is called a non homogeneous linear partial differential eqn's

Its complete sol'n is $z = C.F + P.I$

Method for finding C.F., when $f(D, D')$ can be factorised into linear factors:-

Resolve $f(D, D')$ into linear factors of the form $(D - mD' - \alpha)$. The C.F corresponding to the factor $D - mD' - \alpha$ is $e^{Kx} \phi(y + mx)$.

if $f(D, D') = (D - m_1 D' - \alpha_1)(D - m_2 D' - \alpha_2) \dots (D - m_n D' - \alpha_n)$

$$C.F = e^{K_1 x} f_1(y + m_1 x) + e^{K_2 x} f_2(y + m_2 x) + \dots + e^{K_n x} f_n(y + m_n x)$$

In case of repeated factors

If $f(D, D') = (D - mD' - \alpha)^r$ then

$$C.F = e^{Kx} f_1(y + mx) + x e^{Kx} f_2(y + mx)$$

If $f(D, D') = (D - mD' - \alpha)^3$ then

$$C.F = e^{Kx} f_1(y + mx) + x e^{Kx} f_2(y + mx) + x^2 e^{Kx} f_3(y + mx)$$

Methods for finding P.I.

Let the given eqn be $f(D, D')z = Q(x, y)$.

$$\text{Then } P.I = \frac{1}{f(D, D')} Q(x, y)$$

Case (i) when $Q(x, y) = e^{ax+by}$ and $f(a, b) \neq 0$

$$P.I = \frac{1}{f(a, b)} e^{ax+by}$$

ie replace D by a & D' by b .

(35) Note If $f(a,b)=0$ Then it is a case of failure

in this case $P.I = x \cdot \frac{1}{\frac{\partial}{\partial D} [f(D,D)]} e^{ax+by}$ (or) $y \cdot \frac{1}{\frac{\partial}{\partial D} [f(D,D)]} e^{ax+by}$

Case (ii) when $Q(x,y) = \sin(ax+by)$ or $\cos(ax+by)$

Then $P.I = \frac{1}{f(D,D)} \sin(ax+by)$

put $D^x = -(a^x)$, $D^y = -(b^y)$, $DD^1 = -(ab)$ providing

the denominator is non-zero

Note If the denominator is zero then

$P.I = x \cdot \frac{1}{\frac{\partial}{\partial D} [f(D,D)]} \sin(ax+by)$ or $y \cdot \frac{1}{\frac{\partial}{\partial D} [f(D,D)]} \sin(ax+by)$

use the above conditions providing denominator is non zero, if the denominator is zero then

$P.I = x^2 \cdot \frac{1}{\frac{\partial^2}{\partial D^2} [f(D,D)]} \sin(ax+by)$ or $y^2 \cdot \frac{1}{\frac{\partial^2}{\partial D^2} [f(D,D)]} \sin(ax+by)$

Case (iii) when $Q(x,y) = x^m y^n$ where m and n are positive integers. $P.I = \frac{1}{f(D,D)} x^m y^n = [f(D,D)]^{-1} x^m y^n$

which can be evaluated after expanding $[f(D,D)]^{-1}$ in ascending powers of $\frac{D}{D}$ when $m > n$ or $\frac{D}{D}$ when $m < n$

Case (iv) when $Q(x,y) = e^{ax+by} V$ where V is a function of x and y .

$P.I = \frac{1}{f(D,D)} e^{ax+by} V = e^{ax+by} \frac{1}{f(D+a, D+b)} V$

which can be evaluated by using the previous case.

(36) problems

① solve $(D-D^1-1)(D-D^1-2)z = e^{2x-y}$

Here $f(D,D^1) = (D-D^1-1)(D-D^1-2)$, $Q(x,y) = e^{2x-y}$

Compare each factor of $f(D,D^1)$ with $D-mD^1-\alpha$, we have

$$m_1 = 1, \alpha_1 = 1, \quad m_2 = 1, \alpha_2 = 2$$

$$C.F = e^{\alpha_1 x} f_1(y+m_1, 1) + e^{\alpha_2 x} f_2(y+m_2, 1)$$

$$= e^x f_1(y+1) + e^{2x} f_2(y+1)$$

$$P.I = \frac{1}{f(D,D^1)} Q(x,y)$$

$$= \frac{1}{(D-D^1-1)(D-D^1-2)} e^{2x-y}$$

put $D=2, D^1=1$, we have

$$= \frac{1}{(2+1)(2+1-2)} e^{2x-y} = \frac{1}{2} e^{2x-y}$$

\therefore Complete soln $z = C.F + P.I$

$$\Rightarrow z = e^x f_1(y+1) + e^{2x} f_2(y+1) + \frac{1}{2} e^{2x-y}$$

② solve $(D^2 - DD^1 + D^1 - 1)z = \cos(x+2y)$

Clearly this is non homogeneous D.E since

Here $f(D,D^1)$ have different order term $(D^2 \quad DD^1 \quad D^1 \quad 1)$

Here $f(D,D^1) = D^2 - DD^1 + D^1 - 1$, $Q(x,y) = \cos(x+2y)$

$$\Rightarrow f(D,D^1) = D^2 - DD^1 + D^1 - 1$$

$$= (D+1)(D-1) - D^1(D-1)$$

$$= (D-1)(D+1-D^1)$$

$= (D - 0 \cdot D^1 - 1)(D - D^1 + 1)$ are product of two

different linear factor compare with $D-mD^1-\alpha$,

we have

(37)

$$C.F = e^{\alpha_1 x} f_1(y+m_1 x) + e^{\alpha_2 x} f_2(y+m_2 x)$$

Here $m_1=0, \alpha_1=1, m_2=1, \alpha_2=1$

$$C.F = e^x f_1(y) + e^x f_2(y+x)$$

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{D^2 - DD' + D' - 1} \text{Col}(x+2y) \quad [\text{Col}(ax+by) \text{ Model } \textcircled{2}]$$

put $D^2 = -(a^2) = -1, DD' = -(ab) = -2$

$$P.I = \frac{1}{-1 + 2 + D' - 1} \text{Col}(x+2y) \quad \frac{1}{D'} \rightarrow \int dy$$

$$= \frac{1}{D'} \text{Col}(x+2y) = \int \text{Col}(x+2y) dy$$

(~~x~~ constant)

$$P.I = \frac{\sin(x+2y)}{2}$$

$$\therefore \text{Soln } z = C.F + P.I$$

$$z = e^x f_1(y) + e^x f_2(y+x) + \frac{1}{2} \sin(x+2y)$$

(3) solve $[D^2 - D'^2 - 3D + 3D'] z = xy$

Sol: Here $f(D,D') = D^2 - D'^2 - 3D + 3D', Q(x,y) = xy$

Here $f(D,D')$ is a non homogeneous D.E in D, D' .

$$f(D,D') = (D+D')(D-D') - 3(D-D')$$

$$= (D-D')[D+D'-3] \text{ are product of two}$$

different linear factors Compare with $D-m_1 \alpha$

Here $m_1=1, \alpha_1=0; m_2=-1, \alpha_2=3$.

$$C.F = e^{\alpha_1 x} f_1(y+m_1 x) + e^{\alpha_2 x} f_2(y+m_2 x)$$

$$= e^{0x} f_1(y+x) + e^{3x} f_2(y-x)$$

$$C.F = f_1(y+x) + e^{3x} f_2(y-x)$$

(38)

$$P.D = \frac{1}{f(D, D')} Q(x, y)$$

$$= \frac{1}{(D-D')(D+D'-3)} xy$$

$$= \frac{1}{D \left[1 - \frac{D'}{D} \right] \left\{ -3 \left(1 - \frac{D+D'}{3} \right) \right\}} xy \quad \left(\text{Here } x^m y^n \right)$$

\$m=1, n=1\$, both are same so expressing \$\frac{D'}{D}\$ or \$\frac{D}{D}\$

$$= \frac{1}{3D} \left(1 - \frac{D'}{D} \right)^{-1} \left[1 - \left(\frac{D+D'}{3} \right) \right]^{-1} xy$$

$$= \frac{1}{3D} \left(1 + \frac{D'}{D} + \dots \right) \left[1 + \frac{D+D'}{3} + \left(\frac{D+D'}{3} \right)^2 + \dots \right] xy$$

$$= \frac{1}{3D} \left[1 + \frac{D'}{D} + \dots \right] \left[1 + \frac{D+D'}{3} + \frac{D^2}{9} + \frac{2DD'}{9} + \dots \right] xy$$

$$= \frac{1}{3D} \left[1 + \frac{D+D'}{3} + \frac{2DD'}{3} + \frac{D'}{D} + \frac{D'}{D} \left(\frac{D+D'}{3} \right) + \frac{D'}{D} \left(\frac{2DD'}{3} \right) \right] xy$$

\$\because D^2(x)=0, D^2(y)=0\$
do neglect second and higher order derivatives of \$D, D'\$

$$= \frac{1}{3D} \left[1 + \frac{D}{3} + \frac{D'}{3} + \frac{2DD'}{9} + \frac{D'}{D} + \frac{D}{3} + \frac{D'}{3} + \frac{2DD'}{9} \right] xy$$

$$= \frac{1}{3D} \left[1 + \frac{D}{3} + \frac{2D'}{3} + \frac{D'}{D} + \frac{2DD'}{9} \right] xy$$

$$= \frac{1}{3D} \left[xy + \frac{1}{3}y + \frac{2}{3}x + \frac{x^2}{2} + \frac{2}{9}x \right]$$

$$= \frac{1}{3} \left[\frac{x^2}{2}y + \frac{1}{3}yx + \frac{2}{3} \cdot \frac{x^2}{2} + \frac{x^3}{6} + \frac{2}{9}x \right]$$

$$P.D = \frac{1}{3} \left[\frac{x^2}{2}y + \frac{xy}{3} + \frac{x^2}{3} + \frac{x^3}{6} + \frac{2}{9}x \right]$$

$$DD(xy) = D(x) D(y) = 1$$

$$\frac{D}{D}(xy) = \frac{1}{D}(x) D(y) = \int x dx = \frac{x^2}{2}$$

Complete solution is \$Q = C.F + P.D\$

$$\therefore Q = f_1(y+x) + e^{3x} f_2(y-x) - \frac{1}{3} \left(\frac{x^2}{2}y + \frac{xy}{3} + \frac{x^2}{3} + \frac{x^3}{6} + \frac{2}{9}x \right)$$

39) (4) Solve $(D-3D'-2)^2 z = 2e^{2x} \tan(y+3x)$

Sol: Here $f(D, D') = (D-3D'-2)^2$ are product of two repeated linear factors. Compare with $(D-mD'-a)^n$
Here $m=3, a=2$

$$\begin{aligned} \text{C.F.} &= e^{\alpha x} f_1(y+mx) + x e^{\alpha x} f_2(y+mx) \\ &= e^{2x} f_1(y+3x) + x e^{2x} f_2(y+3x) \end{aligned}$$

Given $Q(x, y) = 2e^{2x} \tan(y+3x)$

$$\begin{aligned} \text{P.I.} &= \frac{1}{f(D, D')} Q(x, y) \\ &= \frac{1}{(D-3D'-2)^2} 2e^{2x} \tan(y+3x) \end{aligned}$$

put $D = D+2, D' = D'+0$ (using model (4))

$$= 2e^{2x} \frac{1}{(D+2-3D')^2} \tan(y+3x) \quad \left(e^{ax+by} Q(x, y) \right)$$

$$= 2e^{2x} \left[\frac{1}{(D-3D')^2} \tan(y+3x) \right] \quad \left(\text{Compare with } \tan(ax+by) \right)$$

Now this eqn. becomes to Homogeneous D.E.

Here $f(D, D') = (D-3D')^2$,

put $D = a=3, D' = b=1, n = \text{order} = 2$

$f(D, D') = f(3, 1) = (3-3)^2 = 0$, Case of failure

$$\text{P.I.} = 2e^{2x} \times \frac{1}{\frac{2(D-3D')^2}{2D}}$$

$$= \cancel{2} e^{2x} \times \frac{1}{\cancel{2}(D-3D')} \tan(y+3x)$$

put $D = a=3, D' = b=1, D-3D' = 3-3=0$

again Case of failure.

(40)

$$P.I = x e^{2x} \cdot x \cdot \frac{1}{\frac{\partial}{\partial x}(D-3D)} \tan(x+3y)$$

$$P.I = x^2 e^{2x} \cdot \tan(x+3y)$$

$$\text{C.I is } z = C.F + P.I$$

$$\Rightarrow z = e^{2x} f_1(y+3x) + x e^{2x} f_2(y+3x) + x^2 e^{2x} \tan(x+3y)$$

$$(5) (D^2 - DD' - 2D) z = \sin(3x+4y)$$

$$\begin{aligned} f(D, D') &= D^2 - DD' - 2D, \quad Q(x, y) = \sin(3x+4y) \\ &= \cancel{D^2 - 2DD' + DD' - 2D} = \cancel{D^2 + DD' - 2DD' - 2D} \\ &= \cancel{D^2 - 2D - 2DD' + DD'} = \cancel{D(D+D')} = 2D \end{aligned}$$

$$= D(D-2)$$

$$f(D, D') = D^2 - DD' - 2D$$

$$= D[D - D' - 2]$$

$$= (D - 0 \cdot D' - 0)(D - D' - 2) \quad \text{Compare with } D - mD' - \alpha$$

$$\text{Here } m_1 = 0, \alpha_1 = 0, \quad m_2 = 1, \alpha_2 = 2$$

$$C.F = e^{\alpha_1 x} f_1(y+m_1 x) + e^{\alpha_2 x} f_2(y+m_2 x)$$

$$= e^{0x} f_1(y+0x) + e^{2x} f_2(y+x)$$

$$C.F = f_1(y) + e^{2x} f_2(y+x)$$

$$P.I = \frac{1}{f(D, D')} Q(x, y)$$

$$= \frac{1}{D^2 - DD' - 2D} \sin(3x+4y) \quad \left(\sin(ax+by), \quad a=3, b=4 \right)$$

$$\text{put } D^2 = -(3^2) = -9, \quad DD' = -(ab) = -12$$

$$P.I = \frac{1}{-9+12-2D} \sin(3x+4y)$$

$$= \frac{1}{3-2D} \frac{a(3+2D)}{(3+2D)} \sin(3x+4y)$$

(51)

$$P \cdot I = \frac{(3+2D)}{9-4D^2} \sin(3x+4y)$$

$$\text{put } D^2 = -(a^2) = -(3^2) = -9$$

$$= \frac{(3+2D)}{9-4(-9)} \sin(3x+4y)$$

$$\Rightarrow P \cdot I = \frac{3 \sin(3x+4y) + 2D[\sin(3x+4y)]}{45}$$

$$= \frac{3 \sin(3x+4y) + 6 \cos(3x+4y)}{45}$$

$$P \cdot I = \frac{1}{15} [\sin(3x+4y) + 2 \cos(3x+4y)] \left[\begin{aligned} D(\sin(3x+4y)) &= \frac{d}{dx} \sin(3x+4y) \\ &= 3 \cos(3x+4y) \end{aligned} \right]$$

$$\text{G.S.S } z = C.F + P \cdot I$$

$$\Rightarrow z = f_1(y) + e^{2x} f_2(y+x) + \frac{1}{15} [\sin(3x+4y) + 2 \cos(3x+4y)]$$

$$(6) \quad r-s+p=1$$

$$\text{sol:} \quad \text{w.k.T } r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad p = \frac{\partial z}{\partial x}$$

Symbolic form of the given D.E is

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} = 1$$

$$\Rightarrow (D^2 - D D' + D)z = 1$$

$$\text{Here } f(D, D') = D^2 - D D' + D, \quad Q(x, y) = 1 = e^{0x+0y}$$

$$\Rightarrow f(D, D') = D[D - D' + 1]$$

$= (D - 0D' + 0)(D - D' + 1)$ are two different linear

factors. Compare with $(D - mD' - \alpha)$, we have

$$m_1 = 0, \alpha_1 = 0; \quad m_2 = 1, \alpha_2 = -1$$

$$C.F = e^{\alpha_1 x} f_1(y + m_1 x) + e^{\alpha_2 x} f_2(y + m_2 x)$$

$$\Rightarrow C.F = e^{0x} f_1(y) + e^{-x} f_2(y+x)$$

$$C.F = f_1(y) + e^{-x} f_2(y+x)$$

(42)

$$P \cdot I = \frac{1}{f(D, D')} Q(x, y)$$

$$= \frac{1}{D^2 - DD' + D} e^{0x+0y} \quad (e^{ax+by} \text{ Model ①})$$

put $D=0, D'=0$, we have

$$P \cdot I = \frac{1}{0} e^{0x+0y}, \text{ it is case of failure}$$

$$= x \cdot \frac{1}{\frac{\partial}{\partial D}(D^2 - DD' + D)} e^{0x+0y}$$

$$= x \cdot \frac{1}{(2D - D' + 1)} e^{0x+0y}$$

put $D=0, D'=0$, we have

$$P \cdot I = x \cdot e^{0x+0y} = x.$$

$$C.S. \text{ 2} = C.F + P \cdot I$$

$$= f_1(y) + e^{-x} f_2(y+x) + x.$$

$$(7) (D+D'-1)(D+2D'-3) z = 4 + 3x + 6y$$

Sol:- $f(D, D') = (D+D'-1)(D+2D'-3), \quad Q(x, y) = 4 + 3x + 6y$

$\downarrow \qquad \qquad \downarrow$
 $m_1 = -1, \alpha_1 = 1 \quad m_2 = -2, \alpha_2 = 3 \quad \text{different}$

$$C.F = e^{\alpha_1 x} f_1(y+m_1 x) + e^{\alpha_2 x} f_2(y+m_2 x)$$

$$C.F = e^x f_1(y-x) + e^{3x} f_2(y-2x)$$

(43)

$$\text{Now } \frac{1}{f(D, D')} e^{0x+0y} = \frac{1}{(D+D'-1)(D+2D'-3)} e^{0x+0y}$$

put $D=0, D'=0$, we have,

$$= \frac{1}{3} e^{0x+0y} = \frac{1}{3}.$$

$$\frac{1}{f(D, D')} x = \frac{1}{(D+D'-1)(D+2D'-3)} x$$

$$= \frac{1}{(-1) [1-(D+D')] (-3) \left[1-\left(\frac{D+2D'}{3}\right)\right]} x$$

$$= \frac{1}{3} [1-(D+D')]^{-1} \left[1-\left(\frac{D+2D'}{3}\right)\right]^{-1} x.$$

$$= \frac{1}{3} [1+(D+D')+(D+D')^2+\dots] \left[1+\frac{1}{3}(D+2D')+\frac{1}{9}(D+2D')^2+\dots\right] x$$

$$= \frac{1}{3} [1+D+\cancel{D'}] \left[1+\frac{1}{3}D+\frac{2}{3}\cancel{D'}\right] x.$$

$$= \frac{1}{3} [1+D] \left[1+\frac{1}{3}D\right] x \quad (\because D'(x)=0; \frac{\partial}{\partial y}(x)=0)$$

$$= \frac{1}{3} \left[1+\frac{1}{3}D+D+\frac{1}{3}D^2\right] x \quad (D^2(x)=0)$$

$$= \frac{1}{3} \left(1+\frac{4}{3}D\right) x = \frac{1}{3} \left(x+\frac{4}{3}\right)$$

$$\frac{1}{f(D, D')} y = \frac{1}{(D+D'-1)(D+2D'-3)} (y)$$

$$= \frac{1}{(-1) [1-(D+D')] (-3) \left[1-\left(\frac{D+2D'}{3}\right)\right]} (y)$$

$$= \frac{1}{3} [1-(D+D')]^{-1} \left[1-\left(\frac{D+2D'}{3}\right)\right]^{-1} (y)$$

$$= \frac{1}{3} [1+\cancel{D}+D'] \left[1+\frac{1}{3}\cancel{D}+\frac{2}{3}D'\right] y \quad (D(y)=0)$$

$$= \frac{1}{3} (1+D')(1+\frac{2}{3}D') y$$

$$= \frac{1}{3} \left(1+\frac{2}{3}D'+D'+\frac{2}{3}D'^2\right) y \quad D''(y)=0$$

$$(14) \frac{1}{f(D,D')} (y) = \frac{1}{3} \left(1 + \frac{5D'}{3}\right) y = \frac{1}{3} \left(y + \frac{5}{3}\right)$$

Sub this in (1), we have

$$P.I = 4\left(\frac{1}{3}\right) + \beta \cdot \frac{1}{3} \left(x + \frac{4}{3}\right) + \gamma \cdot \frac{1}{3} \left(y + \frac{5}{3}\right)$$

$$= \frac{4}{3} + x + \frac{4}{3} + 2y + \frac{10}{3}$$

$$P.I = x + 2y + \frac{18}{3} = x + 2y + 6$$

$$\therefore G.S \text{ is } \delta = C.F + P.I$$

$$= e^x f_1(y-x) + e^{3x} f_2(y+2x) + x + 2y + 6$$

$$(8) (D-3D'-2)^3 \delta = 6 e^{2x} \sin(3x+y)$$

Sol: Here $f(D,D') = (D-3D'-2)^3$, repeated 3 times
 \downarrow
 $m=3, \alpha=2$

$$C.F = e^{\alpha x} f_1(y+mx) + x e^{\alpha x} f_2(y+mx) + x^2 e^{\alpha x} f_3(y+mx)$$

$$\therefore C.F = e^{2x} f_1(y+3x) + x e^{2x} f_2(y+3x) + x^2 e^{2x} f_3(y+3x)$$

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{(D-3D'-2)^3} 6 e^{2x} \sin(3x+y) \left(e^{ax+by} v \right)$$

$$\text{put } D = D + \omega, D' = D' + b, \omega = 2, b = 0$$

$$= 6 e^{2x} \cdot \frac{1}{(D+2-3D'-2)^3} \sin(3x+y)$$

$$= 6 e^{2x} \frac{1}{(D-3D')^3} \sin(3x+y), \text{ This is Homogeneous}$$

Here $\omega=3, b=1, n=\text{order}=3$ D.E of the form $\frac{1}{f(D,D')} \sin(ax+by)$.

$$f_1(D,D') = f_1(3,1) = (3-3)^3 = 0$$

ie. an+by combination fails

Case of failure,

$$P.I = 6 e^{2x} \cdot x \cdot \frac{1}{\frac{\partial}{\partial D} (D-3D')^3} \sin(3x+y)$$

(45)

$$P.I = \frac{2}{6} x e^{2x} \frac{1}{(D-3D)^2} \sin(3x+y)$$

$$\text{put } D=3, D'=1, n=2$$

$$(D-3D)^2 = (3-3)^2 = 0, \text{ Case of failure.}$$

$$P.I = 2 x e^{2x} \left\{ x \frac{1}{\frac{\partial}{\partial D} (D-3D)^2} \sin(3x+y) \right\}$$

$$= \frac{2}{3} x^2 e^{2x} \frac{1}{(D-3D)} \sin(3x+y)$$

$$\text{put } D=\omega=3, D'=b=1, n=1$$

$$D-3D'=3-3=0, \text{ again Case of failure.}$$

$$P.I = x^2 e^{2x} \left\{ x \cdot \frac{1}{\frac{\partial}{\partial D} (D-3D)} \sin(3x+y) \right\}$$

$$P.I = x^3 e^{2x} \sin(3x+y)$$

$$G.S. \quad z = C.F + P.I$$

$$= e^{2x} f_1(y+3x) + x e^{2x} f_2(y+3x) + x^2 e^{2x} f_3(y+3x) + x^3 e^{2x} \sin(3x+y)$$

W.W

$$(2) \quad x-t + p-q=0 \quad A) \quad z = f_1(y+x) + e^x f_2(y-x)$$

$$(2) \quad (D+D'-1)(D+2D'-2)z=0 \quad A) \quad z = e^x f_1(y-x) + e^{2x} f_2(y-2x)$$

$$(3) \quad (D^2 - D'^2 - 3D + 3D')z = e^{x-2y}$$

$$A) \quad z = f_1(y+x) + e^{3x} f_2(y-x) - \frac{1}{12} e^{x-2y}$$

$$(4) \quad (D^2 - DD' + D'-1)z = \sin(x+2y)$$

$$A) \quad z = e^x f_1(y) + e^x f_2(y+x) - \frac{1}{2} \cos(x+2y)$$

$$(5) \quad (D-D'-1)(D-D'-2)z = e^{3x-y} + x$$

$$A) \quad z = e^x f_1(y+x) + e^{2x} f_2(y+x) + \frac{1}{6} e^{3x-y} + \frac{1}{2} (x + \frac{3}{2})$$

(46)

Classification of second order P.D.E

The second order linear P.D.E

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + F(u) = 0$$

where A, B, C, D, E are real constants is said to be

- (i) Hyperbolic if $B^2 - 4AC > 0$
 (ii) parabolic if $B^2 - 4AC = 0$
 (iii) Elliptic if $B^2 - 4AC < 0$

for example.

- (i) The eqn $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$ (Special Case of wave eqn)
 is hyperbolic since $A=1, B=0, C=-1$,

$$B^2 - 4AC = 0 - 4(1)(-1) = 4 > 0$$

- (ii) The eqn $\frac{\partial u}{\partial t} = c \frac{\partial^2 u}{\partial x^2}$ (heat eqn) is parabolic

$$\text{since } A=c, B=0, C=0 \text{ and } B^2 - 4AC = 0$$

- (iii) The eqn $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (two dimensional Laplace eqn) is elliptic, since $A=1, B=0, C=1$

$$B^2 - 4AC = -4 < 0.$$

Applications of PDE.Method of Separation of variables :-

where we have a partial differential eqn involving two independent variables say x and y , we seek a solution in the form $X(x)Y(y)$ and write down various types of solutions. The following example will explain the method.

(47) problems on method of separation of variables

① solve $y^3 \frac{\partial z}{\partial x} + x^2 \frac{\partial z}{\partial y} = 0$ — (1)

Soln let $z = x(x) \gamma(y)$ be the soln of (1)

$$\frac{\partial z}{\partial x} = x'(x) \gamma(y), \quad \frac{\partial z}{\partial y} = x(x) \gamma'(y)$$

sub these in (1), we have

$$y^3 x'(x) \gamma(y) + x^2 x(x) \gamma'(y) = 0$$

$$\Rightarrow y^3 x'(x) \gamma(y) = -x^2 x(x) \gamma'(y) = 0$$

$$\Rightarrow \frac{x'(x)}{x^2 x(x)} = - \frac{\gamma'(y)}{y^3 \gamma(y)}$$

In the above L.H.S is a function of x and R.H.S is a function of y and these are equal for all values of x and y . This is possible if and only if each is equal to the same constant (say) λ . This λ is called Separation Constant

$$\therefore \text{we have } \frac{x'(x)}{x^2 x(x)} = \frac{-\gamma'(y)}{y^3 \gamma(y)} = \lambda \quad \text{--- (2)}$$

from (2), we get the two ordinary differential equations

$$x'(x) = \lambda x^2 x(x) \quad \& \quad \gamma'(y) = -\lambda y^3 \gamma(y)$$

$$\Rightarrow \frac{x'(x)}{x^2 x(x)} = \lambda x^2 x(x)$$

$$\Rightarrow \frac{dx}{dx} = \lambda x^2 x$$

$$\Rightarrow \frac{dx}{x} = \lambda x^2 dx$$

Integrate on B-S

$$\int \frac{dx}{x} = \lambda \int x^2 dx$$

$$\Rightarrow \log x = \lambda \frac{x^3}{3} + \log C_1$$

$$\Rightarrow \log \left(\frac{x}{C_1} \right) = \lambda \frac{x^3}{3}$$

$$\frac{dy}{dy} = -\lambda y^3 \gamma$$

$$\Rightarrow \frac{dy}{y} = -\lambda y^3 dy, \text{ Integrate on B-S}$$

$$\Rightarrow \log y = -\lambda \frac{y^4}{4} + \log C_2$$

$$\Rightarrow \log y - \log C_2 = -\lambda \frac{y^4}{4}$$

$$\Rightarrow \log \left(\frac{y}{C_2} \right) = -\lambda \frac{y^4}{4}$$

$$\Rightarrow \frac{y}{C_2} = e^{-\lambda \frac{y^4}{4}}$$

(48)

$$\Rightarrow \frac{x}{c_1} = e^{\lambda \frac{x^3}{3}}, \quad \frac{y}{c_2} = e^{-\lambda \frac{y^4}{4}}$$

$$\Rightarrow x = c_1 e^{\lambda \frac{x^3}{3}}, \quad y = c_2 e^{-\lambda \frac{y^4}{4}}$$

\therefore soln of (1) is given by $z = x(x)y(y)$

$$\Rightarrow z = c_1 e^{\lambda \frac{x^3}{3}} \cdot c_2 e^{-\lambda \frac{y^4}{4}}$$

$$z = c e^{\lambda \left(\frac{x^3}{3} - \frac{y^4}{4} \right)}, \quad c = c_1 c_2$$

is an arbitrary constant

(2) Solve by the method of separation of variables

$$u_t = 2u_x + u \quad \text{where } u(x, 0) = 6e^{-3x}$$

Sol: Given P.D.E $\frac{\partial u}{\partial t} = 2 \frac{\partial u}{\partial x} + u$ — (1)

Let the soln of eqn (1) by method of separation of variables is

$$u = x(x)T(t)$$

$$\Rightarrow \frac{\partial u}{\partial x} = x' T, \quad \frac{\partial u}{\partial t} = x T'$$

sub these in (1), we have

$$x' T = 2 x T' + x T$$

$$\Rightarrow x' T = x [2 T' + T]$$

$$\Rightarrow \frac{x'}{x} = \frac{2 T' + T}{T} = \lambda \quad (\text{separation constant})$$

Now $\frac{x'}{x} = \lambda$

$$\Rightarrow x' = \lambda x$$

$$\Rightarrow \frac{dx}{dx} = \lambda x$$

$$\Rightarrow \frac{dx}{x} = \lambda dx$$

Variable-separable,

Integrate on B-S

$$\frac{2 T' + T}{T} = \lambda$$

$$2 T' + T = \lambda T$$

$$\Rightarrow 2 T' = (\lambda - 1) T$$

$$\Rightarrow 2 \frac{dT}{dT} = (\lambda - 1) T$$

$$\Rightarrow \frac{dT}{T} = \frac{(\lambda - 1)}{2} dt$$

V-S, Integrate on B-S

(49)

$$\int \frac{dx}{x} = \lambda \int dt$$

$$\Rightarrow \log x = \lambda t + \log c_1$$

$$\Rightarrow \log\left(\frac{x}{c_1}\right) = \lambda t$$

$$\Rightarrow \frac{x}{c_1} = e^{\lambda t}$$

$$\Rightarrow \boxed{x(t) = c_1 e^{\lambda t}}$$

$$\int \frac{dT}{T} = \left(\frac{\lambda-1}{2}\right) \int dt$$

$$\Rightarrow \log T = \left(\frac{\lambda-1}{2}\right)t + \log c_2$$

$$\Rightarrow \log T - \log c_2 = \left(\frac{\lambda-1}{2}\right)t$$

$$\Rightarrow \log\left(\frac{T}{c_2}\right) = \left(\frac{\lambda-1}{2}\right)t$$

$$\Rightarrow \frac{T}{c_2} = e^{\left(\frac{\lambda-1}{2}\right)t}$$

$$\Rightarrow \boxed{T(t) = c_2 e^{\left(\frac{\lambda-1}{2}\right)t}}$$

$$\therefore u(x,t) = x(t)T(t)$$

$$\Rightarrow u(x,t) = c_1 e^{\lambda x} \cdot c_2 e^{\left(\frac{\lambda-1}{2}\right)t}$$

$$\Rightarrow u(x,t) = c e^{\lambda x + \left(\frac{\lambda-1}{2}\right)t} \quad (\because c_1 \cdot c_2 = c)$$

$$\text{but given } u(x,0) = 6 e^{-3x}$$

$$\therefore u(x,0) = c e^{\lambda x}$$

$$\Rightarrow 6 e^{-3x} = c e^{\lambda x}$$

Comparing like coefficients on B.S, we have.

$$\boxed{c=6, \lambda=-3}$$

Sub c, λ values in (2) we get the sol'n of (1)

$$\boxed{u(x,t) = 6 e^{-3x-2t}}$$

(3) Using method of separation of variables solve
 $u_{xt} = e^t \cos x$ with $u(x,0)=0$ and $u(0,t)=0$

Soln Given P.D.E $\frac{\partial^2 u}{\partial x \partial t} = e^t \cos x$ (1)

Let the sol'n of eqn (1) using method of separation of variables is $u = X(x)T(t)$

(50)

$$\Rightarrow \frac{\partial^2 u}{\partial x \partial t} = x' T'$$

sub this in (1), we have

$$x' T' = e^{-t} \cos x \quad \text{where } x' = \frac{dx}{dt}, \quad T' = \frac{dT}{dt}$$

$$\Rightarrow \frac{x'}{\cos x} = \frac{e^{-t}}{T'} = 1 \quad (\text{separation constant})$$

$$\frac{x'}{\cos x} = 1$$

$$\Rightarrow x' = \cos x$$

$$\Rightarrow \frac{dx}{\cos x} = 1$$

$$\Rightarrow dx = \cos x$$

Integrate on both sides

$$\Rightarrow x = \sin x + C_1$$

$$\frac{e^{-t}}{T'} = 1$$

$$\Rightarrow T' = e^{-t}$$

$$\Rightarrow T' = \frac{1}{T} e^{-t} \quad (\text{v-s, } \int \text{on both sides})$$

$$\Rightarrow \frac{dT}{T} = \frac{1}{T} e^{-t}$$

$$\Rightarrow dT = \frac{1}{T} e^{-t} dt$$

$$\Rightarrow T = -\frac{1}{T} e^{-t} + C_2$$

sub $x(t)$ & $T(t)$ values in (2), we have,

$$u(x,t) = [\sin x + C_1] \left[-\frac{1}{T} e^{-t} + C_2 \right] \quad \text{--- (3)}$$

but given $u(x,0) = 0$ & $u(0,t) = 0$.

Now $u(x,0) = 0$, from (3)

$$\Rightarrow 0 = u(x,0) = (\sin x + C_1) \left(-\frac{1}{T} + C_2 \right)$$

$$\Rightarrow 0 = (\sin x + C_1) \left(-\frac{1}{T} + C_2 \right)$$

$$\Rightarrow -\frac{1}{T} + C_2 = 0 \Rightarrow C_2 = \frac{1}{T} \quad (\because \sin x + C_1 \neq 0)$$

Now $u(0,t) = 0$, from (3)

$$u(0,t) = C_1 \left[-\frac{1}{T} e^{-t} + C_2 \right]$$

$$\Rightarrow 0 = C_1 \left[-\frac{1}{T} e^{-t} + C_2 \right] \Rightarrow C_1 = 0$$

sub C_1, C_2 values in (3) we get the solution

$$u(x,t) = \sin x \left[-\frac{1}{T} e^{-t} + \frac{1}{T} \right] = \sin x (1 - e^{-t})$$